MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Series

> Q&A: February 3

Next: Ross § 15

Week 5:

- Homework 4 (due Sunday, February 7)
- Quiz 3 (Wednesday, February 3) - Lectures 8-9

Sequences $\left|\frac{s_{n+1}}{s_n}\right|$ and $\sqrt[n]{|s_n|}$

Thm 12.2 Let (s_n) be a sequence, $\forall n\ (s_n \neq 0)$. Then

Proof. If $\ell = 0$, then $\ell \leq \beta$. Assume that $\ell > 0$.

Take any $0 < \ell_1 < \ell$. Then by Thm 9.11 (i)

Therefore,

$\Rightarrow$

Note that $(\tilde{u}_k)$ is increasing, so $\forall\, k > N$

Now

So $\forall\ \ell_1 \in (0,1)$ $\Rightarrow$ β is an upper bound for $(0, \ell) \Rightarrow$

Sequences $\left|\frac{s_{n+1}}{s_n}\right|$ and $\sqrt[n]{|s_n|}$

Corollary 12.3

If $\lim\limits_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ exists, and $\lim\limits_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = L$, then

Example

Let (a_n) be a sequence such that $\forall n\in\mathbb{N}$ $a_n>0$.

Suppose that (a_n) converges, $\lim\limits_{n\to\infty} a_n = a$. Then

Proof. Denote $s_n := a_1\cdot\dots\cdot a_n$. Then

By Corollary 12.3

Series

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

For $p, q \in \mathbb{N}$, $p<q$ we denote $a_p + a_{p+1} + \cdots + a_q$ by

Def 14.1 (Infinite series) We call the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

an (infinite) series. a_n is called the n-th term of the series.

Def 14.2 (Convergent series)

We call the sum the (n-th) partial sum of the series.

If the sequence (s_n) of partial sums converges, we say that the series $\sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} s_n = s$, then we call s the sum of the series $\sum_{n\geq 1}^{\infty} a_n$, and write it as

Series

If $\lim_{n\to\infty} S_n = +\infty \ (-\infty)$, we say that $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty \ (-\infty)$

and we write

We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely (is absolutely convergent)

if the series

Remark An infinite series can be viewed as a particular type of a sequence, $S_n = a_1 + a_2 + \cdots + a_n$

so we can use all the relevant results.

For example, if $\forall n \ a_n \geq 0$, then S_n is increasing.

Partial sums of $\sum_{n=1}^{\infty} |a_n|$ form an increasing sequence.

Use the criteria on convergence for partial sums etc.

Important examples

8. Let $a, r \in \mathbb{R}$. Then is called the geometric series.

If $|r|<1$, then $\sum_{n=0}^{\infty} a r^n =$

Proof Denote $S_k = \sum_{n=0}^{k} a r^n =$

Note that $r(1+r+\cdots+r^k) =$, so

9. Let $p>0$. Then converges iff

Proof ($p=2$). $S_k := \sum_{n=1}^{k} \frac{1}{n^2}$. 1 (S_k) is increasing

2 (S_k) is bounded

For any $n \geq 2$, so

1 + 2 + Thm 10.2

Cauchy criterion

Def 14.3 We say that $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion if its sequence of partial sums (s_n) is a Cauchy sequence, i.e.

$$\forall \varepsilon > 0 \quad \exists N \quad \forall m, n > N \quad |s_n - s_m| < \varepsilon$$

Thm 14.4 $\sum a_n$ converges $\Leftrightarrow$ $\sum a_n$ satisfies the Cauchy criterion

Proof. Follows from Thm 10.11

Corollary 14.5 (Necessary condition for convergence).

$\sum a_n$ converges $\Rightarrow$

Proof. $\sum a_n$ converges $\overset{\text{Thm 14.4}}{\Longleftrightarrow}$

$\Rightarrow$ $\Leftrightarrow$ $\lim a_n = 0$ ∎

Example

- $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ satisfies the Cauchy criterion

Proof. $\forall$ $k \in \mathbb{N}$, so $\forall$ $n > m \geq 1$

$$\sum_{k=m+1}^{n} \frac{1}{k2^k} \leq$$

Fix $\varepsilon > 0$. By 1.E.2 Therefore

In particular,

- If $|r| \geq 1$, then the sequence (r^n) does not converge to 0

$\Rightarrow$

- Consider , but the series diverges

Comparison test

Thm 14.6 Let (a_n) and (b_n) be two sequence, $\forall n$ $a_n \geq 0$

Then

(i)

(ii)

Proof. (i) Use the Cauchy criterion

Fix $\varepsilon > 0$. By Thm 14.4 . Then

By Thm 14.4 .

(ii) Denote $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$. Then

$\sum_{n=1}^{\infty} a_n = +\infty \Leftrightarrow$