MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Properties of continuous functions

> Q&A: February 10

Next: Ross § 19

Week 6:

- Homework 5 (due Sunday, February 14)
- Regrades of HW3 (Monday, February 8 - Wednesday, February 10)

The maximum-value theorem

Def. 18.7 Let f be a function and let $A \subset \mathrm{dom}(f)$. f is called bounded on A if $\exists M > 0 \ \forall x \in A \ \ |f(x)| \le M$

Thm 18.1 Let f be a function, $[a,b] \subset \mathrm{dom}(f)$, f is continuous on $[a,b]$

Then (i) f is bounded on $[a,b]$

(ii) $\exists x_0, y_0 \in [a,b]$ s.t. $\forall x \in [a,b] \ \big(f(x_0) \le f(x) \le f(y_0)\big)$

min value max value

Proof (i) Suppose that f is not bounded on $[a,b]$

$\Rightarrow \forall n \in \mathbb{N} \ \exists x_n \in [a,b]$ s.t $|f(x_n)| > n \quad (*)$

1. (x_n) is bounded $\overset{T\,11.5}{\Rightarrow} \exists (x_{n_k})$ s.t. (x_{n_k}) converges
2. $\forall k \quad a \le x_{n_k} \le b \overset{T.9.11}{\Longrightarrow} \lim x_{n_k} =: \tilde{x} \in [a,b]$
3. $\tilde{x} \in [a,b]$, f cont. on $[a,b] \overset{T.17.3}{\Longrightarrow} |f|$ is cont. at $\tilde{x}$

$\Rightarrow \lim |f(x_{n_k})| = |f(\tilde{x})| \Rightarrow \exists N \ \forall k > N \ \big(|f(x_{n_k})| < |f(\tilde{x})| + 1\big)$

contradiction to $(*)$.

The maximum-value theorem

<u>Proof</u> (ii) Denote $M := \sup\{f(x) : x \in [a,b]\}$. By (i), $M < +\infty$

1. $M = \sup\{f(x) : x \in [a,b]\} \Rightarrow \forall n \ \exists x_n \in [a,b] \ (M - \frac{1}{n} < f(x_n) \leq M)$
2. $\forall n \ (M - \frac{1}{n} < f(x_n) \leq M) \overset{T.9.11}{\Rightarrow} \lim f(x_n) = M$
3. $\forall n \ (x_n \in [a,b]) \overset{T\,11.5}{\Rightarrow} \exists \ (x_{n_k}) \ \exists \ y_0 \in [a,b]$ s.t. $\lim x_{n_k} = y_0$
4. $y_0 \in [a,b] \Rightarrow f$ is continuous at $y_0 \Rightarrow \lim f(x_{n_k}) = f(y_0)$

   $\Rightarrow$ by T 11.3 $f(y_0) = M \Rightarrow \forall x \in [a,b] \ (f(x) \leq f(y_0))$

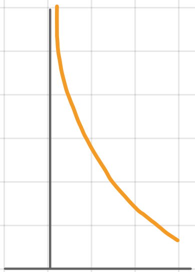
(Exercise: prove that $\exists \ x_0 \in [a,b]$ s.t. $\forall x \in [a,b] \ (f(x_0) \leq f(x))$)

■

<u>Examples</u>

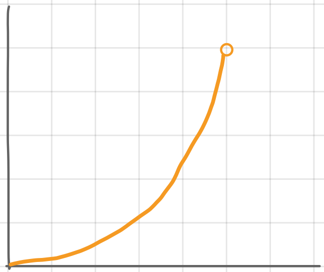
1) $f(x) = \frac{1}{x}$

- continuous on $(0,1]$
- unbounded on $(0,1]$



2) $f(x) = x^2$

- cont on $[0,1)$
- no maximum on $[0,1)$



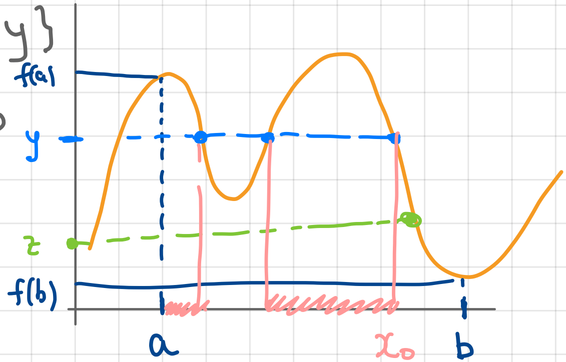
Intermediate value theorem

Thm 18.2 Let f be continuous on the interval $I \subset \mathbb{R}$. Let $a, b \in I$ s.t $a < b$. Then

(i) $f(a) < f(b)$ and $y \in (f(a), f(b)) \Rightarrow \exists\, x \in (a,b)$ s.t. $f(x) = y$

(ii) $f(a) > f(b)$ and $y \in (f(b), f(a)) \Rightarrow \exists\, x \in (a,b)$ s.t. $f(x) = y$

Proof (ii) Consider $S = \{x \in [a,b] : f(x) > y\}$



1. $a \in S$, $\sup S \geq a$, $\sup(S) \leq \sup[a,b] = b$

   $\Rightarrow \sup S =: x_0 \in [a,b]$

2. $\forall n \in \mathbb{N}\ \exists\, s_n \in S$ s.t. $x_0 - \frac{1}{n} < s_n \leq x_0$

Then $\lim s_n = x_0 \wedge f(s_n) > y \wedge f$ cont at $x_0 \Rightarrow f(x_0) = \lim f(s_n) \geq y$

3. Define $t_n := \min\{x_0 + \frac{1}{n}, b\}$. $\forall n\ t_n \in [a,b] \setminus S \Rightarrow \forall n\ f(t_n) \leq y$

   $\lim t_n = x_0 \Rightarrow f(x_0) = \lim f(t_n) \leq y \overset{②+③}{\Rightarrow} f(x_0) = y \Rightarrow x_0 \in (a,b)$ $\blacksquare$

Image of an interval

Cor. 18.3 Let f be continuous on the interval I. Then $f(I) = \{ f(x) : x \in I \}$ is an interval or a single point.

Proof If $\forall\, x \in I$ $f(x) = y_0$, then $f(I) = y_0$.

① Let $y_1 < y_2 \in f(I)$. Then $\exists\, x_1, x_2 \in I$ s.t. $f(x_1) = y_1$, $f(x_2) = y_2$

Let $y \in (y_1, y_2)$.

- If $x_1 < x_2$, then by Thm 18.2 (i) $\exists\, x_0 \in (x_1, x_2) \subset I$ s.t. $f(x_0) = y$

  $\Rightarrow y \in f(I)$

- If $x_2 < x_1$, then by Thm 18.2 (ii) $\exists\, x_0 \in (x_2, x_1) \subset I$ s.t. $f(x_0) = y$

  $\Rightarrow y \in f(I)$.

② Let $\inf f(I) < \sup f(I)$. Then $\forall\, y \in (\inf f(I), \sup f(I))$ $\exists\, y_1, y_2 \in f(I)$

s.t. $y_1 < y < y_2 \Rightarrow y \in f(I) \Rightarrow f(I) \supset (\inf f(I), \sup f(I))$ ∎

Examples

1) $\sin : (0, 2\pi) \to \mathbb{R}$

$\sin((0,2\pi)) \subset [-1,1]$

$\sin\left(\frac{\pi}{2}\right) = 1$

$\sin\left(\frac{3\pi}{2}\right) = -1$

$\Rightarrow [-1,1] \subset \sin((0,2\pi))$

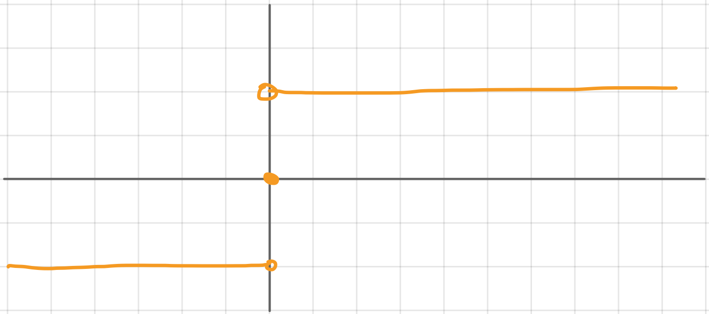
2) $f : [-1,1] \to \mathbb{R}$, $f(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

$f([-1,1]) = \{-1, 0, 1\}$

$-1 = f\left(-\frac{1}{2}\right) < f(0) = 0$

But $\forall \; y \in (-1, 0)$

$\{x \in (-\frac{1}{2}, 0) : f(x) = y\} = \emptyset$



Continuity of strictly increasing functions.

Def 18.8 Function f is called

(strictly) increasing if $x<y \Rightarrow f(x)\le f(y)$ $(f(x)<f(y))$

(strictly) decreasing if $x<y \Rightarrow f(x)\ge f(y)$ $(f(x)>f(y))$

Thm 18.5 Let g be strictly increasing function on interval J.

If $g(J)$ is an interval, then g is continuous of J

Proof. Let $x_0\in J$, $x_0>\inf J$, $x_0<\sup J$. Then $g(x_0)>\inf g(J)$

$g(x_0)<\sup g(J) \Rightarrow \exists\, \varepsilon_0>0$ s.t. $(g(x_0)-\varepsilon_0,\ g(x_0)+\varepsilon_0)\subset g(J)$

Verify the ε-δ definition of continuity. Fix $\varepsilon>0$, $\varepsilon<\varepsilon_0$.

Then $\exists\, x_1, x_2\in J$ s.t. $g(x_1)=g(x_0)-\varepsilon$, $g(x_2)=g(x_0)+\varepsilon$, $x_1<x_0<x_2$

Now, $\forall x\in(x_1,x_2)$ $g(x_1)<g(x)<g(x_2) \Rightarrow |g(x)-g(x_0)|<\varepsilon$

Take $\delta:=\min\{x_0-x_1, x_2-x_0\}$ Then $|x-x_0|<\delta \Rightarrow x\in(x_1,x_2)$

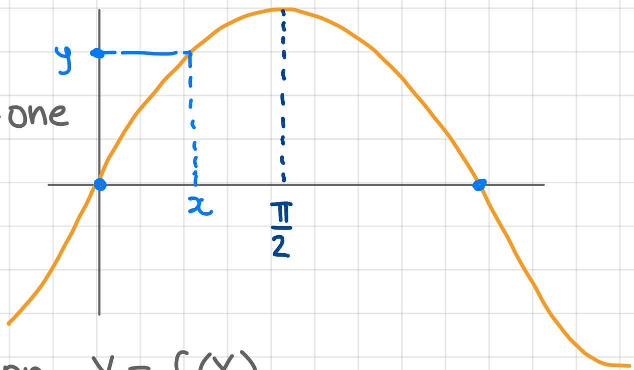
$\Rightarrow |g(x)-g(x_0)|<\varepsilon$ ∎

Inverse function

Def 18.9 Function $f: X \to Y$ is called one-to-one (or bijection)

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

if $f(X) = Y$ and $\forall y \in Y \ \exists! \, x \in X$ s.t. $f(x) = y$



Example $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is one-to-one

$\sin: [0, \pi] \to [0,1]$ is not one-to-one

$\sin(0) = \sin(\pi) = 0$

Def 18.10 Let $f: X \to Y$ be a bijection, $Y = f(X)$.

Then the function $f^{-1}: Y \to X$ given by $\left(f^{-1}(y) = x \Leftrightarrow f(x) = y \right)$

is called the inverse of f. In particular $f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = y$

Example
- $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$, $\sin^{-1} = \arcsin: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $f: [0, +\infty) \to [0, +\infty)$, $f(x) = x^m$, $f^{-1}: [0, +\infty) \to [0, +\infty)$, $f^{-1}(x) = x^{\frac{1}{m}} = \sqrt[m]{x}$
- If f is strictly increasing (decreasing) on X, then $f: X \to f(X)$ is a bijection

Continuity and the inverse function

Thm 18.4 Let f be a continuous strictly increasing function on some interval I. Then $J := f(I)$ is an interval and

$f^{-1}: J \to I$ is continuous and strictly increasing.

Proof ① f^{-1} is strictly increasing: Take $y_1, y_2 \in J$, $y_1 < y_2$

Denote $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Then $f(x_1) = y_1$, $f(x_2) = y_2$

If $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$, contradiction $\Rightarrow x_1 < x_2$

② J is an interval: By Cor. 18.3 J is either an interval or a single point. Since f is strictly increasing, J is an interval

③ ① + ② + Thm 18.5 f^{-1} is continuous on J.

One-to-one continuous functions

Thm 18.6 Let f be a one-to-one continuous function on an interval I. Then f is strictly increasing or strictly decreasing on I.

Proof. ① If $a<b<c$ then either $f(a)<f(b)<f(c)$ or $f(c)<f(b)<f(a)$

Otherwise, $f(b)>\max\{f(a),f(c)\}$ or $f(b)<\min\{f(a),f(c)\}$

If $f(b)>\max\{f(a),f(c)\}$, choose $y\in(\max\{f(a),f(c)\},f(b))$

Then by Thm 18.2 $\exists x_1\in(a,b)$ s.t. $f(x_1)=y$, $\exists x_2\in(b,c)$ s.t. $f(x_2)=y$

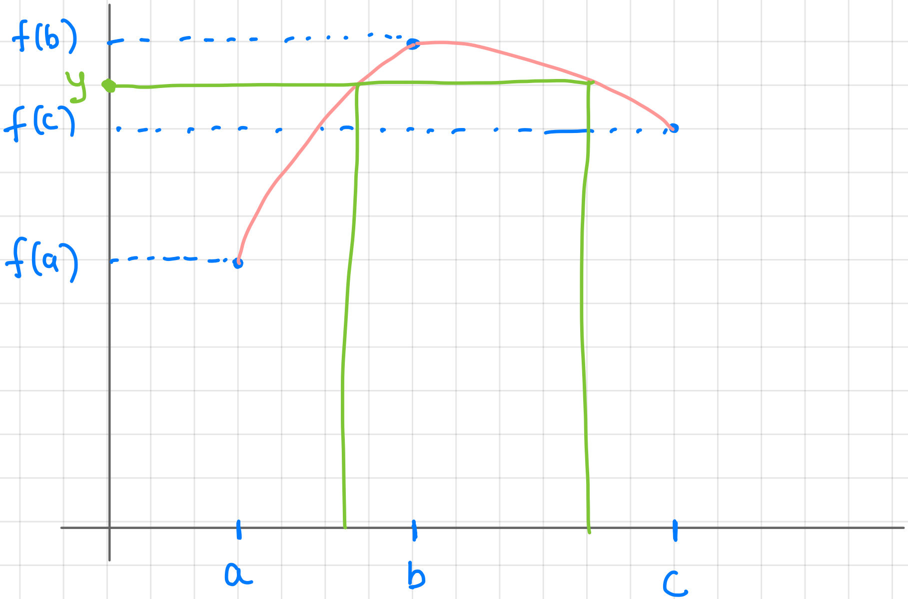
$\Rightarrow$ contradiction. Similarly when $f(b)<\min\{f(a),f(c)\}$.

② Take any $a_0<b_0$. If $f(a_0)<f(b_0)$, then f is increasing on I.

$$x<a_0<y<b_0<z \overset{①}{\Rightarrow} f(a_0)<f(y)<f(b_0) \quad \Big| \quad \begin{array}{l} \Rightarrow f(x)<f(a_0)<f(y) \\ ① \\ \Rightarrow f(y)<f(b_0)<f(z) \end{array}$$

$$\Rightarrow \forall x_1<x_2 \quad (f(x_1)<f(x_2))$$

③ Similarly, if $f(a_0)>f(b_0)$, then f is decreasing. ∎


f(b)
y
f(c)
f(a)
a
b
c