MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Uniform continuity

> Q&A: February 17

Next: Ross § 20

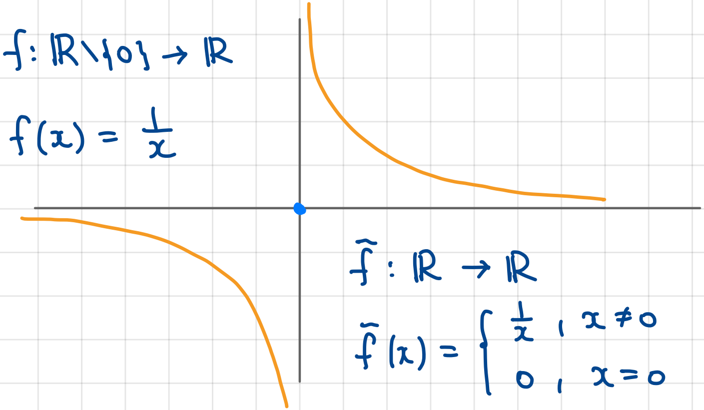
Week 7:

- Homework 6 (due Sunday, February 21)
- Quiz 4 (Wednesday, February 17)

Extension of a function

$f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) = \frac{1}{x}$



$\tilde{f}: \mathbb{R} \to \mathbb{R}$

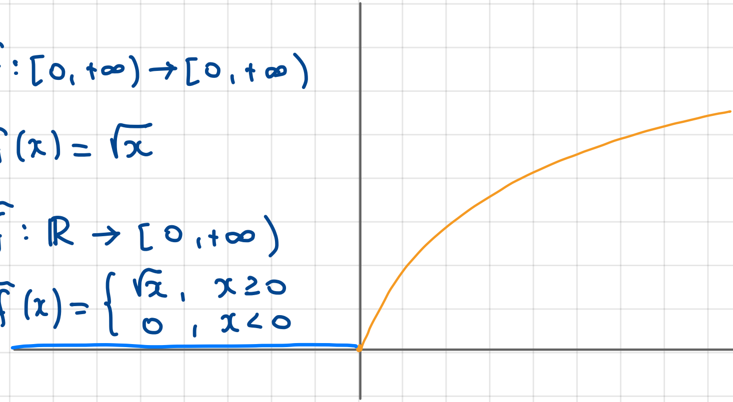
$$\tilde{f}(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f: [0, +\infty) \to [0, +\infty)$

$f(x) = \sqrt{x}$

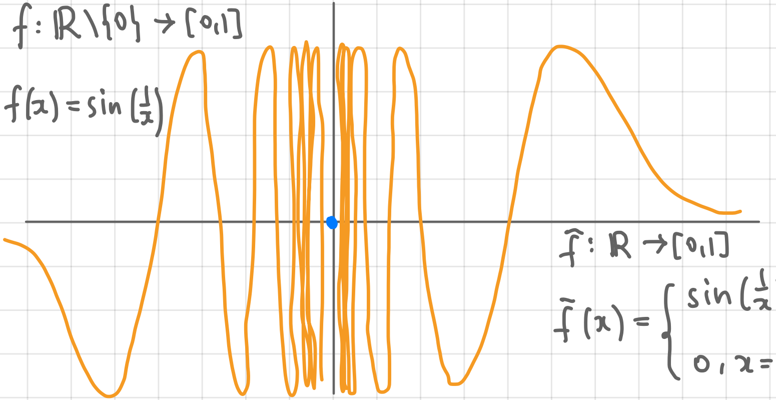
$\tilde{f}: \mathbb{R} \to [0, +\infty)$

$$\tilde{f}(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$f: \mathbb{R}\setminus\{0\} \to [0,1]$

$f(x) = \sin\left(\frac{1}{x}\right)$

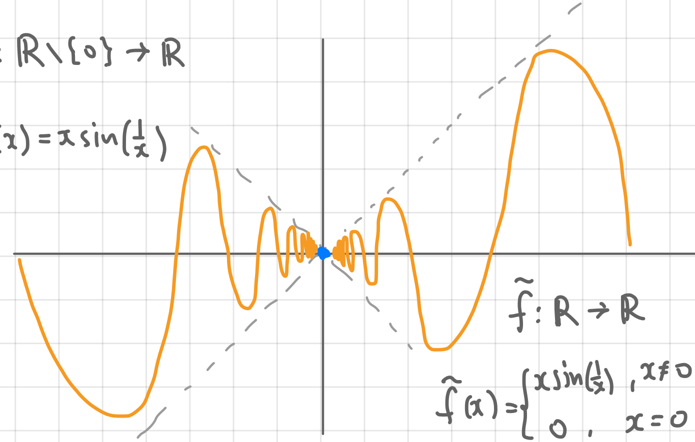


$\tilde{f}: \mathbb{R} \to [0,1]$

$$\tilde{f}(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) = x\sin\left(\frac{1}{x}\right)$



$\tilde{f}: \mathbb{R} \to \mathbb{R}$

$$\tilde{f}(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Def. 19.7. Let f and $\tilde{f}$ be two functions s.t. $\mathrm{dom}(f) \subset \mathrm{dom}(\tilde{f})$

We say that $\tilde{f}$ is an extention of f if $\forall x \in \mathrm{dom}(f)$ $\tilde{f}(x) = f(x)$

Continuous extention

Thm 19.5 A real-valued function f on (a,b) is uniformly continuous on (a,b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a,b]$.

Proof $(\Leftarrow)$ $\tilde{f}$ is cont. on $[a,b] \overset{T\,19.2}{\Rightarrow} \tilde{f}$ is unif. cont on $[a,b]$

$\Rightarrow \tilde{f}$ is unif. cont. on $(a,b) \Rightarrow f$ is unif. cont. on (a,b).

$(\Rightarrow)$ Suppose f is unif. cont. on (a,b). How to define $\tilde{f}(a)$ and $\tilde{f}(b)$?

① Let (s_n) be a sequence, $s_n \in (a,b)$, $\lim s_n = a$. Then $(f(s_n))$ converges

(s_n) converges $\Rightarrow (s_n)$ is a Cauchy sequence $\overset{T\,19.4}{\Rightarrow} (f(s_n))$ is a Cauchy sequence

② Let (s_n) and (t_n) be two sequences, $\forall n$ $s_n, t_n \in (a,b)$, $\lim s_n = \lim t_n = a$

Take $(u_n) = (s_1, t_1, s_2, t_2, \dots)$ Then $u_n \in (a,b)$, $\lim u_n = a \overset{①}{\Rightarrow} \lim (f(u_n)) =: L$

$\overset{T\,11.3}{\Rightarrow}$ $\lim f(s_n) = \lim f(t_n) = L =: \tilde{f}(a)$

③ $\tilde{f}$ is continuous at a (follows from Lemma 19.8). ∎

Continuous extension

Lemma 19.8 (Ex. 17.15) Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at $x_0 \in \mathrm{dom}(f)$ iff for any sequence (x_n) in $\mathrm{dom}(f) \setminus \{x_0\}$ converging to x_0, we have $\lim f(x_n) = f(x_0)$

Proof ($\Rightarrow$) Trivial

($\Leftarrow$) Let (s_n) be a sequence in $\mathrm{dom}(f)$, $\lim s_n = x_0$.

(i) $\{n : s_n \neq x_0\}$ is finite $\Rightarrow \exists N \; \forall n > N \; s_n = x_0 \Rightarrow \forall n > N \; f(s_n) = f(x_0)$

(ii) $\{n : s_n \neq x_0\}$ is infinite. Let (s_{n_k}) be a subsequence of (s_n) obtained by removing all terms equal to x_0. Then (s_{n_k}) is a sequence in $\mathrm{dom}(f) \setminus \{x_0\}$, $\lim s_{n_k} = x_0 \Rightarrow \lim f(s_{n_k}) = f(x_0)$

Fix $\varepsilon > 0$. Then $\exists K \; \forall k > K \; |f(s_{n_k}) - f(x_0)| < \varepsilon \Rightarrow \forall n > n_K$
$|f(s_n) - f(x_0)| < \varepsilon$

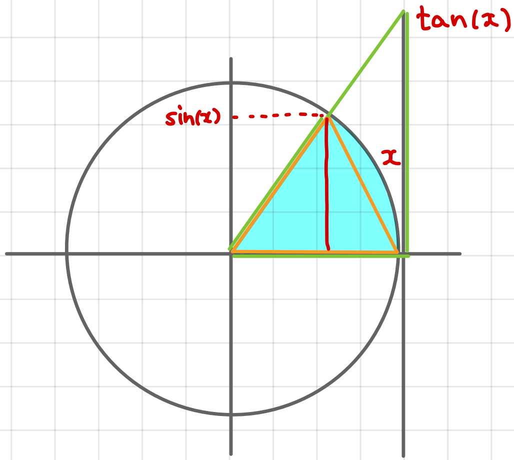
Examples

1. $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $[-n,n]\setminus\{0\}$, but not uniformly continuous on $[-n,n]\setminus\{0\}$ (cannot be continuously extended to $[-n,n]$)

IE 10. $f(x) = \frac{\sin x}{x}$ is continuous on $[-n,n]\setminus\{0\}$

$$\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$ is continuous on $[-n,n] \Rightarrow f$ is unif. cont. on $[-n,n]\setminus\{0\}$

Proof: Area ($\triangle$) $\leq$ Area ($\triangle$) $\leq$ Area ($\triangle$)



$0 < |x| < \frac{\pi}{2}$: $\quad \frac{1}{2}|\sin x| < \frac{1}{2}|x| < \frac{1}{2}|\tan x| = \frac{1}{2}\frac{|\sin x|}{\cos x}$

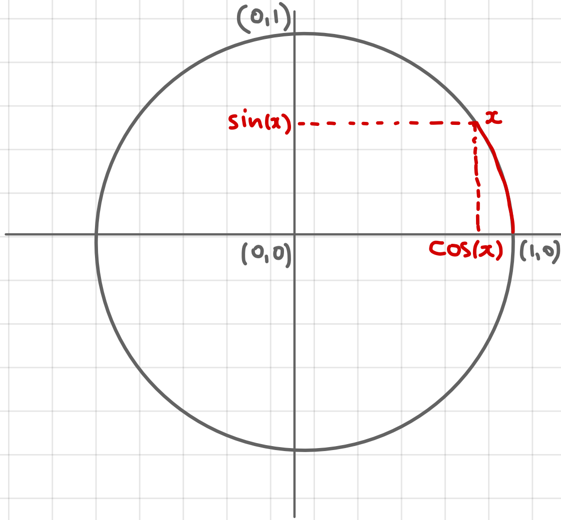
$\cos x < \frac{\sin x}{x} < 1 \Rightarrow 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\frac{x}{2} < 2\cdot\frac{x^2}{4}$

We want to show that $\hat{f}$ is cont. at $x=0$.

Fix $\varepsilon > 0$. Let (s_n) be a sequence in $[-n,n]\setminus\{0\}$,

$\lim s_n = 0 \Rightarrow \exists N \; \forall n > N \; |s_n| < \sqrt{\varepsilon} \Rightarrow \forall n > N \; \left|1 - \frac{\sin s_n}{s_n}\right| < \frac{s_n^2}{2} < \varepsilon$ ∎

Definition of some functions



sin, cos, tan, cotan

sin, cos are continuous on $\mathbb{R}$

x^n, $x \in \mathbb{R}$, $n \in \mathbb{N}$

x^n is continuous on $\mathbb{R}$ for any $n \in \mathbb{N}$

x^n is a bijection from $[0,+\infty)$ to $[0,+\infty)$,

we denote the inverse by $\sqrt[n]{x} = x^{\frac{1}{n}}$, $x \geq 0$, $n \in \mathbb{N}$

$\forall\ a > 0\ \forall\ m, n \in \mathbb{N}$ $\quad (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m =: a^{\frac{m}{n}}$

Let $b > 0$, (q_n) s.t. $q_n \in \mathbb{Q} \cap (0,+\infty)$, $q_n < q_{n+1}$, $\lim q_n = b$

For $a > 1$ (a^{q_n}) is increasing and bounded above $\Rightarrow$ $\lim\limits_{n\to\infty} a^{q_n} =: a^b > 0$

Define $\left(\frac{1}{a}\right)^b = \frac{1}{a^b} = a^{-b}$, $a^0 = 1$

Satisfies usual properties: $a^{b_1} a^{b_2} = a^{b_1+b_2}$, $a_1^b a_2^b = (a_1 a_2)^b$, ...

Definition of some functions

For any $a > 1$ the function $f : \mathbb{R} \to (0, \infty)$, $f(x) = a^x$ is strictly increasing, we denote the inverse by $\log_a x$

Similarly for $a \in (0,1)$, a^x is strictly decreasing.

Usual properties hold: $\log_a x_1 + \log_a x_2 = \log_a (x_1 x_2)$, ...

Special notation: $\log_e x = \log x = \ln x$

Example of a proof: $a^{b_1} a^{b_2} = a^{b_1 + b_2}$

1. If $b_1 = m_1$, $b_2 = m_2$, $m_1, m_2 \in \mathbb{N}$, then $a^{m_1} \cdot a^{m_2} = a^{m_1 + m_2}$
2. If $b = \frac{1}{n}$, $a_1, a_2 \in (0, +\infty)$, then $a_1^{\frac{1}{n}} \cdot a_2^{\frac{1}{n}} = (a_1 a_2)^{\frac{1}{n}}$
3. If $b_1 = \frac{m_1}{n}$, $b_2 = \frac{m_2}{n}$, then $a^{b_1} a^{b_2} = a^{b_1 + b_2}$
4. Let (s_n), (t_n), $\lim s_n = b_1$ $\lim t_n = b_2$, (s_n), (t_n) increasing in $\mathbb{Q}$

   $\forall n$ $a^{s_n} a^{t_n} = a^{s_n + t_n}$, $(s_n + t_n)$ increasing $\Rightarrow \lim a^{s_n} a^{t_n} = \lim a^{s_n + t_n}$