MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Limits of functions

> Q&A: February 19

Next: Ross § 20

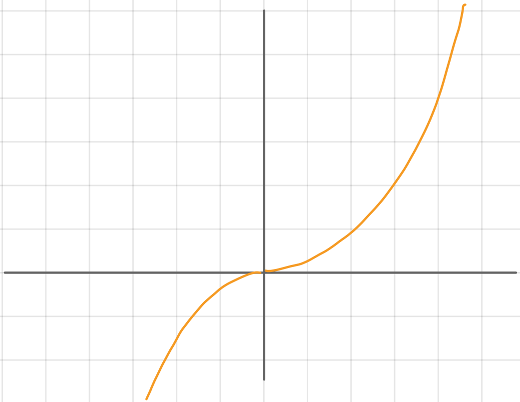
Week 7:

- Homework 6 (due Sunday, February 21)
- Midterm 2 (Wednesday, February 24)

Limit of a Function

Def 17.1 (Continuity). Let f be a real-valued function, $\mathrm{dom}(f) \subset \mathbb{R}$. Function f is continuous at $x_0 \in \mathrm{dom}(f)$ if for any sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$

$$\lim f(x_n) = f(\lim x_n)$$



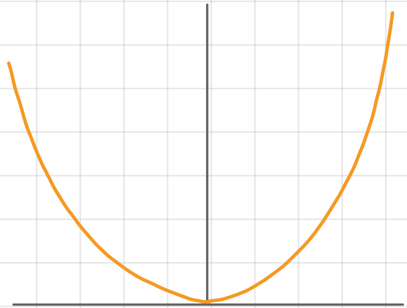
Def 20.1 (Limit of a function)

Let $S \subset \mathbb{R}$, $a, L \in \mathbb{R} \cup \{-\infty, +\infty\}$, suppose that there is a sequence in S for which a is the limit. Let $f: S \to \mathbb{R}$ be a function.

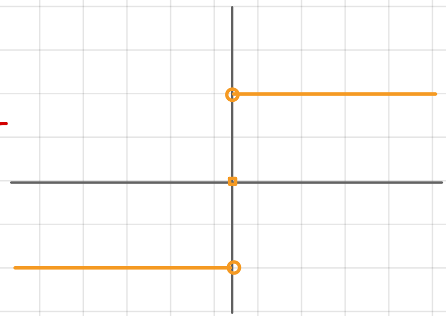
We say that f tends to L as x tends to a along S, or that L is the limit of f as x tends to a along S, if for every sequence (x_n) in S ( ). Notation

Limit of a Function

Definitions 20.3



(a) We say that f tends to L as x tends to a, or that L is the (two-sided) limit of f as x tends to a if $\lim\limits_{S \ni x \to a} f(x) = L$ for $S =$; $\lim\limits_{x \to a} f(x) = L$



(b) L is the right-hand limit of f at a if $\lim\limits_{S \ni x \to a} f(x) = L$ for $S =$ with $c > 0$; $\lim\limits_{x \to a^+} f(x) = L$

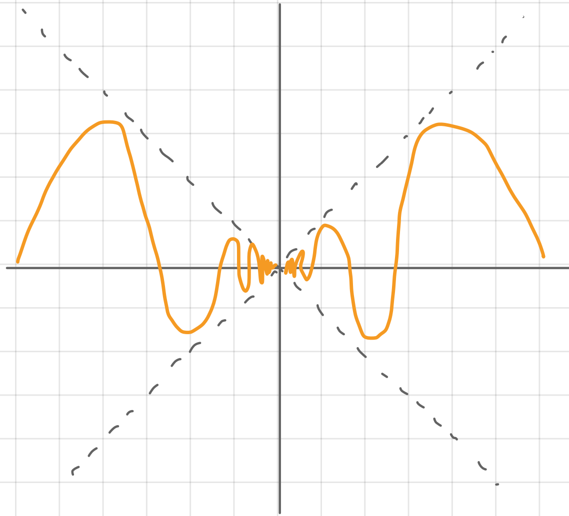
(c) L is the left-hand limit of f at a if $\lim\limits_{S \ni x \to a} f(x) = L$ for $S =$ with $c > 0$; $\lim\limits_{x \to a^-} f(x) = L$

(d) $\lim\limits_{x \to +\infty} f(x) = L \iff \lim\limits_{S \ni x \to +\infty} f(x) = L$ for $S =$, $c \in \mathbb{R}$

$\lim\limits_{x \to -\infty} f(x) = L \iff \lim\limits_{S \ni x \to -\infty} f(x) = L$ for $S =$, $c \in \mathbb{R}$

Examples

1) $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$



Take any $c>0$. Take any sequence (x_n) in $(-c,c)\setminus\{0\}$ s.t. $\lim x_n = 0$. Then $x \mapsto x\sin\left(\frac{1}{x}\right)$ is well-defined for all x_n.

Fix $\varepsilon > 0$,

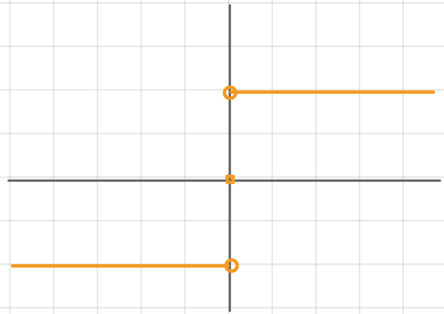
2) $\lim\limits_{x \to +\infty} x \sin\left(\frac{1}{x}\right) = 1$

Take any $c>0$. Take any sequence (x_n) in $(c,+\infty)$, $\lim x_n = +\infty$.

Denote $y_n = \frac{1}{x_n}$. Then by T.9.10

Examples

4) $f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$



$\lim\limits_{x\to 0^+} \operatorname{sgn}(x) = 1$: let (x_n) be a sequence,

$x_n \in (0,1)$, $\lim x_n = 0$. Then

$\lim\limits_{x\to 0} \operatorname{sgn}(x)$ does not exist: Take a sequence

$\lim x_n = 0$, but diverges.

5) $f(x) = \dfrac{x+1}{x-1}$, not defined at $x=1$

$\lim\limits_{x\to 1^+} \dfrac{x+1}{x-1} = +\infty$: take (x_n), $\lim x_n = 1$, $x_n > 1$

Fix $M>0$,

6) If $f: S \to \mathbb{R}$ is continuous at $a \in S$, then $\lim\limits_{S \ni x \to a} f(x) = f(a)$

$\dfrac{x+1}{x-1}$ is continuous at $x=-1$

Limits and arithmetic operations

Thm 20.4 Let f_1 and f_2 be functions for which the limits

$L_1 = \lim_{S \ni x \to a} f_1(x)$ and $L_2 = \lim_{S \ni x \to a} f_2(x)$ exist and are finite. Then

(i) $\lim_{S \ni x \to a} (f_1 + f_2)(x) = L_1 + L_2$ (ii) $\lim_{S \ni x \to a} (f_1 f_2)(x) = L_1 \cdot L_2$

(iii) if $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$, then $\lim_{S \ni x \to a} \frac{f_1}{f_2}(x) = \frac{L_1}{L_2}$

Proof. Follows from Thm. 9.3, 9.4, 9.6.

Take any sequence (x_n) in S that converges to a. Then

$$\lim f_1(x_n) = L_1, \quad \lim f_2(x_n) = L_2.$$ Then

(i) By Thm 9.3 $\lim(f_1(x_n) + f_2(x_n)) = \lim f_1(x_n) + \lim f_2(x_n) = L_1 + L_2$

(ii) By Thm 9.4 $\lim(f_1(x_n) \cdot f_2(x_n)) = \lim f_1(x_n) \cdot \lim f_2(x_n) = L_1 \cdot L_2$

(iii) By Thm 9.6 $\lim \frac{f_1(x_n)}{f_2(x_n)} = \frac{\lim f_1(x_n)}{\lim f_2(x_n)} = \frac{L_1}{L_2}$ ■

Limit of a composition of functions

Thm 20.5

(a) $\lim_{S \ni x \to a} f(x) = L$

(b) g is defined on $\{f(x) : x \in S\} \cup \{L\}$ $\Rightarrow$

(c) g is continuous at L

Proof Let (x_n) be a sequence in S, $\lim x_n = a$.

(a) $\Rightarrow$

(b) + (c) $\Rightarrow$

Example

$f(x) = \sin(x)$, $g(x) = \operatorname{sgn}(x)$ – not continuous at 0. Then

for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad g \circ f(x) =$

Important example 11

(A) Let $a > 1$. Then

Take any sequence (x_n) in $\mathbb{R} \setminus \{0\}$, $\lim x_n = 0$. Fix $\varepsilon > 0$.

① By IE 4

② By IE 4 and Thm 9.5

③ Take ; $\lim x_n = 0 \Rightarrow$

④

(B) Let $a > 1$. Then $x \mapsto a^x$. Take $x_0 \in \mathbb{R}$, take (x_n), $x_n \neq x_0$, $\lim x_n = x_0$. Then

Important example II

(C) $\forall\, a>0$, $x \mapsto a^x$ is continuous on $\mathbb{R}$

If $a \in (0,1)$, then $\forall\, x \in \mathbb{R}$, where

is continuous by (B), is continuous by Thm 17.3

composition $g \circ f(x)$ is continuous (on $\mathbb{R}$) by Thm 17.5

If $a=1$, then $a^x = 1$ $\forall\, x$, continuous.

(D) $\forall\, a>0, a\neq 1$, $x \mapsto \log_a x$ is continuos on $(0,+\infty)$ by Thm 18.4

$x \to a^x$ is strictly increasing $(a>1)$ or strictly decreasing $(a<1)$

and maps $\mathbb{R}$ to $(0,+\infty)$