MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Cauchy sequences

> Q&A: January 29

Next: Ross § 11

Week 4:

- Homework 3 (due Sunday, January 31)
- Midterm 1 on Wednesday, January 27 (lectures 1-7)
- Regrades for HW1: Mon, Jan 25 - Tue, Jan 26 (PST) on Gradescope

Cauchy sequences

Def 7.1. A sequence (s_n) of real numbers is said to converge to the real number s if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ (|s_n - s| < \varepsilon)$$

Def. 10.8 A sequence (s_n) is called a Cauchy sequence if

Examples Fix $\varepsilon > 0$.

1. $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $m, n > N \Rightarrow$

2. $b_n = \frac{(-1)^n}{n}$: $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots$ $m, n > N \Rightarrow$

3. $c_n = 1 + \frac{1}{n}$: $1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ $m, n > N \Rightarrow$

Cauchy sequences

Lemma 10.9 Convergent sequences are Cauchy sequences.

Proof. Suppose $\lim_{n\to\infty} s_n = s \in \mathbb{R}$. Fix $\varepsilon > 0$.

Then

Using the triangle inequality,

Therefore,

Lemma 10.10 Cauchy sequences are bounded.

Proof. Suppose (s_n) is a Cauchy sequence. Then (take $\varepsilon = 1$)

With

Cauchy sequences converge

Thm 10.11 (s_n) converges $\Leftrightarrow$ (s_n)

Proof $(\Rightarrow)$ Lemma 10.9.

$(\Leftarrow)$ Suppose (s_n) is a Cauchy sequence.

By Lemma 10.10 Therefore, by Thm 10.7

it is enough to show that

Denote $u_n = \inf\{s_k : k > n\}$, $v_n = \sup\{s_k : k > n\}$.

Fix $\varepsilon > 0$. Then

Similarly,

Take

Therefore,

Examples

1) Let $a_n = \frac{\cos(1)}{2} + \frac{\cos(2)}{2^2} + \frac{\cos(3)}{2^3} + \cdots + \frac{\cos(n)}{2^n}$. Then

Proof. Fix $\varepsilon > 0$. Then $\forall\ m > n > N$

$|a_m - a_n| =$

$=$

$\leq$

2) Let $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Take $\varepsilon = \frac{1}{2}$. Then $\forall n$

$\Rightarrow$

Asymptotic behavior of sequences

Lemma 10.12 (Exercise 9.12)

Assume that all $s_n \neq 0$ and that $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L \in [0, +\infty)$.

(a) If $L < 1$, then

(b) if $L > 1$, then

Proof. Let $L \in [0,1)$. Fix $\varepsilon > 0$.

Then by Thm 9.11 (i) (Lec 6)

In particular, $|s_{N+2}|$, $|s_{N+3}|$, ..., $|s_{N+k}| <$

Consider the sequence

(i) by Thm 9.2 (Lec 5) and Important example 2 (Lec 6),

(ii) , therefore by Thm 9.11 (ii) Lec 6

Finally,

Example

Exercise 9.13.

$$\lim_{n\to\infty} a^n = \left\{ \right.$$

Proof. Case $|a|<1$: Consider the sequence

Case $a=1$:

Case $a>1$:

Case $a\leq -1$:

Then $\forall N \in \mathbb{N}$ $\cdot$

$\cdot$

Therefore,

Important example 6 (asymptotic growth).

For any $p \in \mathbb{N}$ and any $a > 1$

(exponential sequences grow to ∞ faster than polynomial sequences)

Proof. Denote Then

①

② By Thm 9.4 + ① (applied p-1 times)

$\Rightarrow$

③ By Lemma 10.12 ,

Important example 7 (asymptotic growth).

For any $a > 1$

(factorial grows to ∞ faster than any exponential sequence)

Proof. Denote

①

② By Lemma 10.12,