MATH 142A: Introduction to Analysis

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Today: Subsequences

> Q&A: Jan 29, Feb 1

Next: Ross § 11-12

Week 4:

- Homework 3 (due Sunday, January 31)

Subsequences

$a_n = (-1)^n, n \geq 1$: $-1, 1, -1, 1, -1, 1, -1, 1, \ldots$

$n_k = 2k-1$, $(a_{n_k}) = (-1, -1, -1, -1, \ldots)$; $n_k = 2k$, $(a_{n_k}) = (1, 1, 1, 1, \ldots)$

$b_n = \cos(\frac{\pi n}{2}), n \geq 1$: $(0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots)$

$n_k = 2k-1$, $(b_{n_k}) = (0, 0, 0, 0)$; $n_k = 3k$, $(b_{n_k}) = (0, -1, 0, 1, 0, \ldots)$

$c_n = n, n \geq 1$: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots$ ~~$(2, 1, 4, 3, 6, 5, \ldots)$~~

$(n_k) = (1, 2, 3, 5, 7, 11, 13, \ldots)$, $(c_{n_k}) = (1, 2, 3, 5, 7, 11, 13, \ldots)$

$d_n = \cos(n), n \geq 1$: $\cos(1), \cos(2), \cos(3), \cos(4), \cos(5), \cos(6), \ldots$

$(n_k) = (1, 7, 8, 9, 23, 24, 1002, \ldots)$ $(d_{n_k}) = (\cos(1), \cos(7), \cos(8), \ldots)$

Def 11.1 Let (s_n) be a sequence of real numbers and let $1 \leq n_1 < n_2 < \cdots < n_k < \cdots$ be an increasing sequence of natural numbers. Then $(s_{n_k})_{k=1}^{\infty} = (s_{n_1}, s_{n_2}, s_{n_3}, \ldots)$ is called a subsequence of $(s_n)_{n=1}^{\infty}$.

Subsequences

Thm 11.2 Let (s_n) be a sequence. Let $t \in \mathbb{R}$.

(i) There exists a (monotonic) subsequence of (s_n) converging to t

$$\Leftrightarrow \forall \varepsilon > 0 \text{ the set } \{n \in \mathbb{N} : |s_n - t| < \varepsilon\} \text{ is infinite}$$

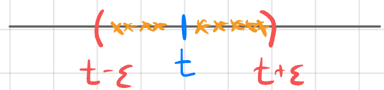
Proof. $(\Rightarrow)$ Exercise.

$(\Leftarrow)$ $\forall \varepsilon > 0$ the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite.

Case 1: the set $\{n : s_n = t\}$ is infinite, take (s_{n_k}) with $s_{n_k} = t$ $\forall k$.

Case 2: $\forall \varepsilon > 0$ the set $\{n : 0 < |s_n - t| < \varepsilon\}$ is infinite.

Either (a) $\forall \varepsilon > 0$ $\{n : t - \varepsilon < s_n < t\}$ is infinite



or (b) $\forall \varepsilon > 0$ $\{n : t < s_n < t + \varepsilon\}$ is infinite

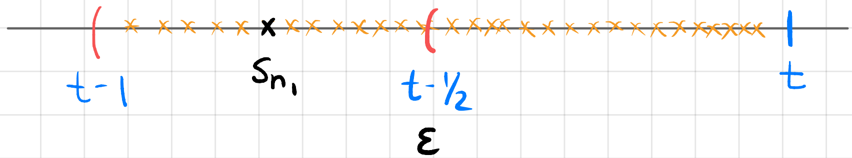
Consider Case 2(a). We want to construct an increasing subsequence that converges to t.

Proof of Thm 11.2 (i)

Suppose that $\forall \varepsilon > 0$ $\{n : t-\varepsilon < S_n < t\}$ is infinite

① Choose n_1 such that

$t-1 < S_{n_1} < t$



② Take $\varepsilon = t - \max\{S_{n_1}, t-\frac{1}{2}\}$, so that $t-\varepsilon = \max\{S_{n_1}, t-\frac{1}{2}\}$

$\varepsilon > 0$ (since $S_{n_1} < t$, and $t - \frac{1}{2} < t$), and thus the set

$S_2 := \{n : \underbrace{\max\{S_{n_1}, t-\frac{1}{2}\}}_{t-\varepsilon} < S_n < t\}$ is infinite.

Choose $n_2 \in S$ such that $n_2 > n_1$. Then $\max\{S_{n_1}, t-\frac{1}{2}\} < S_{n_2} < t$.

⋮

ⓚ Suppose we have numbers $n_1 < n_2 < \cdots < n_{k-1}$ such that

$\forall j \ (\max\{S_{n_{j-1}}, t-\frac{1}{j}\} < S_{n_j} < t)$. Take $\overset{>0}{\varepsilon} = t - \max\{S_{n_{k-1}}, t-\frac{1}{k}\}$

$\{n : t-\varepsilon < S_n < t\}$ is infinite $\Rightarrow \exists\, n_k > n_{k-1}$ s.t. $\max\{S_{n_{k-1}}, t-\frac{1}{k}\} < S_{n_k} < t$

$(S_{n_k})_{k=1}^{\infty}$ is a subsequence of $(S_n)_{n=1}^{\infty}$, and $\forall k$ $t-\frac{1}{k} < S_{n_k} < t \Rightarrow S_{n_k} \xrightarrow[k\to\infty]{} t$ ■

Subsequences

Thm 11.2 Let (s_n) be a sequence.

(ii) (s_n) has a (monotonic) subsequence that diverges to $+\infty$

$\Leftrightarrow$ (s_n) is unbounded above

(iii) (s_n) has a (monotonic) subsequence that diverges to $-\infty$

$\Leftrightarrow$ (s_n) is unbounded below

Proof (ii) $(\Rightarrow)$ Exercise.

$(\Leftarrow)$ Suppose that (s_n) is unbounded above.

1. Let $n_1 = 1$, so that $s_{n_1} = s_1$

2. (s_n) unbounded above $\Rightarrow$ $T_2 := \{n : \max\{s_1, 2\} < s_n\}$ is infinite

choose $n_2 \in T_2$ s.t. $n_2 > n_1$

⋮

k. $T_k := \{n : \max\{s_{n_{k-1}}, k\} < s_n\}$ is infinite, choose $n_k \in T$ s.t. $n_k > n_{k-1}$

Then (s_{n_k}) is a subsequence, $\forall k$ $s_{n_k} > k \Rightarrow \lim_{k\to\infty} s_{n_k} = +\infty$ ∎

Subsequences

Thm 11.3 If (s_n) converges, then any subsequence of (s_n) converges to the same limit.

Proof. Let (s_{n_k}) be a subsequence of (s_n).

① $\forall k \in \mathbb{N} \; (n_k \geq k)$

Proof by induction: $n_1 \geq 1$

if $n_{k-1} \geq k-1$, then $n_k \geq n_{k-1} + 1 \geq k$

② Suppose (s_n) converges to $s \in \mathbb{R}$. Fix $\varepsilon > 0$. Then

(s_{n_k})

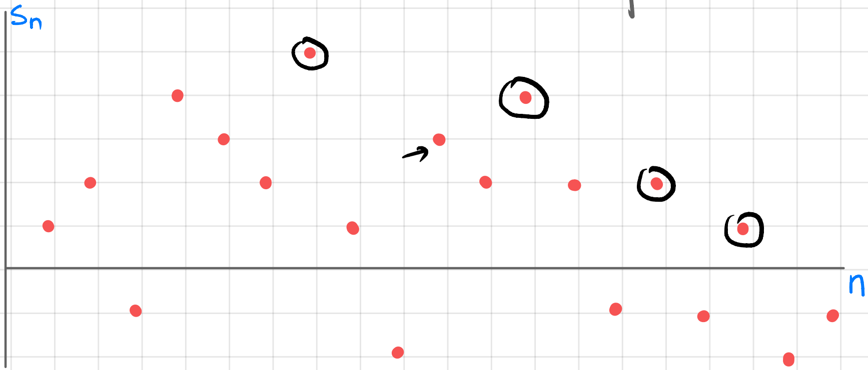
$\exists N \in \mathbb{N} \; \forall n > N \; (|s_n - s| < \varepsilon)$. But since $\forall k \; n_k \geq k$

$\forall k > N \; (n_k > N)$ and thus $(|s_{n_k} - s| < \varepsilon)$

∎

Subsequences

Thm 11.4 Every sequence has a monotonic subsequence.



Proof Let (s_n) be a sequence of real numbers.

We say that s_n is dominant if $\forall\, m>n \; (s_n > s_m)$

Denote $D = \{n : s_n \text{ is dominant}\}$

Case 1: D is infinite. Take $n_1 = \min D, \dots, n_k = \min\{n \in D : n > n_{k-1}\}$

Then $n_1 < n_2 \Rightarrow s_{n_1} > s_{n_2}$, $n_{k-1} < n_k \Rightarrow s_{n_{k-1}} > s_{n_k} \Rightarrow (s_{n_k})$ is decreasing

Case 2: D is finite. Take $n_1 = \max D + 1$ Then s_{n_1} is not dominant

$\Rightarrow \exists\, n_2 > n_1$ s.t. $s_{n_2} \geq s_{n_1}$ Term s_{n_2} is not dominant $\Rightarrow \exists\, n_3 > n_2 \,(s_{n_3} \geq s_{n_2})$

If we have n_{k-1}, then $s_{n_{k-1}}$ is not dominant $\Rightarrow \exists\, n_k > n_{k-1} \,(s_{n_k} \geq s_{n_{k-1}})$

$\Rightarrow (s_{n_k})$ is increasing ∎

Bolzano-Weierstrass Theorem

Thm 11.5 Every bounded sequence has a convergent subsequence.

Proof Let (s_n) be a bounded sequence.

By Thm 11.4 (s_n) has a monotonic subsequence (s_{n_k})

Since (s_n) is bounded, (s_{n_k}) is also bounded.

(s_{n_k}) is monotonic and bounded, therefore by Thm 10.2 $(s_{n_k})_{k=1}^{\infty}$ converges.

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