MATH 142A: Introduction to Analysis

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Today: Continuous functions

> Q&A: February 7

Next: Ross § 18

Week 6:

- Homework 5 (due Sunday, February 13)
- Homework 3 regrades Tuesday, February 8

Functions

Def. (Function) Let X and Y be two sets. We say that there is a function defined on X with values in Y, if via some rule f we associate to each element $x \in X$ an (one) element $y \in Y$. We write $f : X \to Y$, $x \overset{f}{\mapsto} y$ (or $y = f(x)$).

X is called the domain of definition of the function, $\mathrm{dom}(f)$, $y = f(x)$ is called the image of x. $f : [0,1) \to [0,1)$, $x \mapsto x^2$

Remarks 1) We consider real-valued functions ($Y \subset \mathbb{R}$) of one real variable ($X \subset \mathbb{R}$).

2) If $\mathrm{dom}(f)$ is not specified, then it is understood that we take the natural domain: the largest subset of $\mathbb{R}$ which the function is well defined

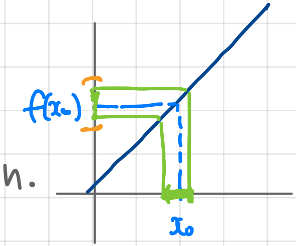
$$\left(\begin{array}{ll} f(x) = \sqrt{x} & \text{means } \mathrm{dom}(f) = [0, +\infty) \\ g(x) = \dfrac{1}{x^2 - x} & \text{means } \mathrm{dom}(g) = \mathbb{R} \setminus \{0, 1\} \end{array} \right)$$

Continuity of a function at a point

Intuitively: Function f is continuous at point $x_0 \in \mathrm{dom}(f)$ if $f(x)$ approaches $f(x_0)$ as x approaches x_0.

Def 17.1 (Continuity). Let f be a real-valued function, $\mathrm{dom}(f) \subset \mathbb{R}$. Function f is continuous at $x_0 \in \mathrm{dom}(f)$ if for any sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$

$$\lim f(x_n) = f(\lim x_n)$$



Def 17.6 (Continuity) Let f be a real-valued function. Function f is continuous at $x_0 \in \mathrm{dom}(f)$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \left(x \in \mathrm{dom}(f) \wedge |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \right) \quad (*)$$

Remark Def 17.1 is called the sequential definition of continuity, Def 17.6 is called the ε-δ definition of continuity.

Equivalence of sequential and ε-δ definitions

Thm 17.2. Definitions 17.1 and 17.6 are equivalent

Proof ($17.1 \Rightarrow 17.6$). Suppose that (*) fails

$$\forall \varepsilon>0 \ \exists \delta>0 \ \left(x\in \mathrm{dom}(f) \wedge |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon \right) \quad (*)$$

This means that

$$\exists \varepsilon>0 \ \forall \delta>0 \ \exists x\in \mathrm{dom}(f) \ \left(|x-x_0|<\delta \wedge |f(x)-f(x_0)|\geq \varepsilon \right)$$

Take $\delta=\frac{1}{n}$: $\exists x_n \in \mathrm{dom}(f) \ \left(|x_n-x_0|<\frac{1}{n} \wedge |f(x_n)-f(x_0)|\geq\varepsilon \right)$

$\Rightarrow \exists (x_n)$ s.t. $\lim x_n = x_0 \wedge \limsup |f(x_n)-f(x_0)|\geq \varepsilon$, contradiction

($\Leftarrow$). Let (x_n) be such that $\lim x_n = x_0$. Fix $\varepsilon>0$. By (*)

$$\exists \delta>0 \ \left(x\in \mathrm{dom}(f) \wedge |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon \right)$$

$\lim\limits_{n\to\infty} x_n = x_0 \Rightarrow \exists N \ \forall n>N \ \left(|x_n-x_0|<\delta \right)$ Therefore

$$\forall n>N \ \left(x_n\in \mathrm{dom}(f) \wedge |x_n-x_0|<\delta \right) \overset{(*)}{\Longrightarrow} \forall n>N \ \left(|f(x_n)-f(x_0)|<\varepsilon \right)$$

$$\Rightarrow \lim f(x_n) = f(x_0) \quad \blacksquare$$

Continuity on a set. Examples

Def Let f be a function, and let $S \subset \mathrm{dom}(f)$.
f is continuous on S if for all $x_0 \in S$ f is continuous at x_0.

Example 1) $f(x) = \frac{2x}{x^2-1}$ is continuous on $\mathbb{R}\setminus\{-1,1\}$

Proof. Let $x_0 \in \mathbb{R}\setminus\{-1,1\}$ and let (x_n) be such that $\forall n\ x_n \notin \{-1,1\}$ and $\lim x_n = x_0$. Then by Thm 9.2, 9.3, 9.6

$$\lim f(x_n) = \lim \frac{2x_n}{x_n^2-1} = \frac{2\lim x_n}{(\lim x_n)^2-1} = \frac{2x_0}{x_0^2-1} = f(x_0)$$

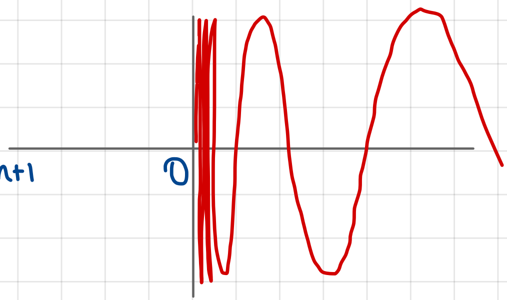
By Def 17.1 f is continuous at x_0 for any $x_0 \in \mathbb{R}\setminus\{-1,1\}$

2) $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = a$. Then for any $a \in \mathbb{R}$ g is not continuous at 0.

Proof Take (x_n) with $x_n = \frac{2}{\pi(2n-1)}$

Then $\lim x_n = 0$ and $g(x_n) = \sin\left(\frac{\pi(2n-1)}{2}\right) = (-1)^{n+1}$

$\Rightarrow \forall a \in \mathbb{R}$ $\lim g(x_n) = a$ fails



Continuity and arithmetic operations

Thm 17.3 Let f be a real-valued function with $\mathrm{dom}(f) \subset \mathbb{R}$.
If f is continuous at $x_0 \in \mathrm{dom}(f)$, then $|f|$ and $k \cdot f$, $k \in \mathbb{R}$, are continuous at x_0.

Proof. Let (x_n) be a sequence in $\mathrm{dom}(f)$ such that $\lim\limits_{n\to\infty} x_n = x_0$.
Then by Thm 9.2 $\lim k \cdot f(x_n) = k \cdot \lim f(x_n) = k \cdot f(x_0)$

Therefore $k \cdot f$ is continuous at x_0.

By the triangle inequality $\big||f(x_n)| - |f(x_0)|\big| \le |f(x_n) - f(x_0)|$

Fix $\varepsilon > 0$. Then $\lim f(x_n) = f(x_0) \Rightarrow \exists N \ \forall n > N \ |f(x_n) - f(x_0)| < \varepsilon$

Then $\forall n > N$ $\big||f(x_n)| - |f(x_0)|\big| \le |f(x_n) - f(x_0)| < \varepsilon$

This means that $\lim\limits_{n\to\infty} |f(x_n)| = |f(x_0)|$, $|f|$ is continuous at x_0.

∎

Continuity and arithmetic operations

Thm 17.4 Let f and g be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then

(i) $f+g$ is continuous at x_0 (ii) $f \cdot g$ is continuous at x_0

(iii) if $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

Proof: Note that if $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, then $(f+g)(x) = f(x) + g(x)$ and $f \cdot g(x) = f(x) \cdot g(x)$ are well-defined. Moreover, if $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $g(x) \neq 0$, then $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is well-defined.

Let (x_n) be a sequence in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ s.t. $\lim x_n = x_0$.

Then $\lim (f(x_n) + g(x_n)) \overset{T.9.3}{=} \lim f(x_n) + \lim g(x_n) = f(x_0) + g(x_0)$, and

$\lim (f(x_n) \cdot g(x_n)) \overset{T9.4}{=} \lim f(x_n) \cdot \lim g(x_n) = f(x_0) \cdot g(x_0)$. If moreover $\forall n\ g(x_n) \neq 0$

then $\lim \frac{f(x_n)}{g(x_n)} \overset{T.9.6}{=} \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{f(x_0)}{g(x_0)}$. [$\mathrm{dom}\left(\frac{f}{g}\right) = \mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) : g(x) \neq 0\}$

■

Continuity of a composition of functions

Let f and g be real-valued functions. If $x \in \mathrm{dom}(f)$ and $f(x) \in \mathrm{dom}(g)$, then we define $g \circ f(x) := g(f(x))$, $\mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$

Thm 17.5 If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof It is given that $x_0 \in \mathrm{dom}(f)$ and $f(x_0) \in \mathrm{dom}(g)$. Let (x_n) be a sequence such that $\forall n \; x_n \in \mathrm{dom}(g \circ f)$ and $\lim x_n = x_0$. Denote $y_n = f(x_n)$, $y_0 = f(x_0)$. Since f is continuous at x_0, $\lim y_n = \lim f(x_n) = f(x_0) = y_0$. Since g is continuous at $f(x_0) = y_0$, we have $\lim g \circ f(x_n) = \lim g(y_n) = g(y_0) = g \circ f(x_0)$. Therefore, $g \circ f$ is continuous at x_0.

Examples

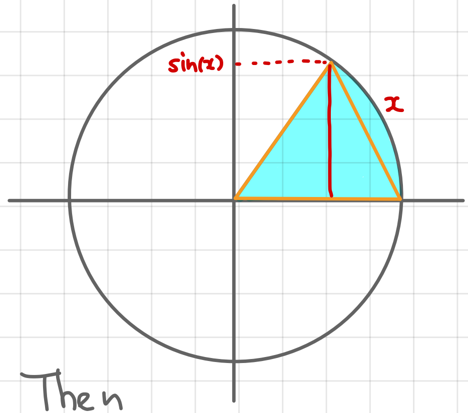
1) $\sin(x)$ is continuous on $\mathbb{R}$

Proof ① Enough to show that $\sin(x)$ is continuous at 0

For any $x_0 \in \mathbb{R}$ and (x_n) with $\lim x_n = x_0$

$$|\sin(x_n) - \sin(x_0)| = \left|2\sin\left(\frac{x_n - x_0}{2}\right)\cos\left(\frac{x_n + x_0}{2}\right)\right| \leq \left|2\sin\left(\frac{x_n - x_0}{2}\right) - 0\right|$$

② Area $(\triangle) \leq$ Area $(\triangle)$



$\Rightarrow \forall x \in [0, \frac{\pi}{2}]$ $\quad \frac{1}{2}\sin(x) \leq \pi \cdot \frac{x}{2\pi} = \frac{x}{2}$

$$\left|\frac{1}{2}\sin(-x)\right| \leq \left|\frac{x}{2}\right|$$

$\Rightarrow \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\quad |\sin(x)| \leq |x|$

③ If $\lim y_n = 0$, then $\exists N\ \forall n > N\ |y_n| \leq \frac{\pi}{2}$. Then

$\forall\ n > N\ \ 0 \leq |\sin(y_n)| \leq |y_n| \overset{T.9.11(ii)}{\Rightarrow} \lim \sin(y_n) = 0$

$\sin 0 = 0$ ∎

Examples

2) $f(x)=\sqrt{x}$ is continuous on $[0,+\infty)$.

1 $\sqrt{x}$ is continuous at 0

Let $\lim x_n = 0$. Fix $\varepsilon>0$. Then $\exists N\ \forall n>N\quad x_n<\varepsilon^2$

$\Rightarrow \forall n>N\quad \sqrt{x_n}<\varepsilon \Rightarrow \lim \sqrt{x_n}=0$

2 Let $x_0\in(0,+\infty)$, (x_n) s.t. $\forall n\ (x_n\in[0,+\infty))$ and $\lim x_n = x_0$

Then $\lim x_n = x_0>0 \overset{T9.11(i)}{\Rightarrow} \exists N_1\ \forall n>N_1\ \left(x_n>\frac{x_0}{2}\right)$

Fix $\varepsilon>0$. Then $\exists N_2\ \forall n>N_2\quad |x_n-x_0|<\sqrt{x_0}\cdot\varepsilon$. Then

$$\forall n>\max\{N_1,N_2\}\quad |f(x_n)-f(x_0)| = |\sqrt{x_n}-\sqrt{x_0}| = \left|\frac{x_n-x_0}{\sqrt{x_n}+\sqrt{x_0}}\right| \le \frac{|x_n-x_0|}{\sqrt{x_0}}<\varepsilon$$

3) $\cos(x)$ is continuous on $\mathbb{R}$. $\cos(x)=\sqrt{1-\sin^2(x)}$, by Thm 17.4

$1-\sin^2(x)$ is continuous on $\mathbb{R}$. Moreover, $\forall x\in\mathbb{R}\quad 1-\sin^2(x)\in[0,1]\subset[0,+\infty)$

$\Rightarrow$ by example 2) and Thm 17.5 $\cos(x)$ is continuous on $\mathbb{R}$.