MATH 142A Introduction to Analysis - FINAL

Winter 2021

March 16, 2021

1 Final Tuesday 8 PM

1.1 Problem 1

1. (15 points) Let (a_n) and (b_n) be two sequences of real numbers such that the sequence (a_n+b_n) is bounded and $\lim a_n = 0$.

   Prove that $\lim a_n b_n = 0$.

1.2 Problem 2

2. (15 points) Let (a_n) be a Cauchy sequence. Prove that the sequence $\sqrt{a_n}$ is also a Cauchy sequence.

1.3 Problem 3

3. (15 points) Determine if the following series converges

$$\sum_{n=1}^{\infty} \frac{n^3\left(\sqrt{2}+(-1)^n\right)^n}{3^n}. \tag{1.1}$$

   Justify your answer.

1.4 Problem 4

4. (15 points) Let function $f : (a, b) \to \mathbb{R}$ be such that

   (i) f is bounded on (a, b);

   (ii) f is continuous on (a, b);

   (iii) f is monotonic on (a, b).

   Prove that f is uniformly continuous on (a, b).

   (Hint. You can use Theorem 19.5.)

1.5 Problem 5

5. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and satisfy

$$f'(x) = \lambda f(x) \tag{1.2}$$

   for some $\lambda > 0$.

   Prove that $f(x) = Ce^{\lambda x}$ for some $C \in \mathbb{R}$.

   (Hint. Consider function $g(x) = f(x)e^{-\lambda x}$ and its derivative.)

1.6 Problem 6

6. (15 points) Compute the limit

$$\lim_{x \to 1} x^{\frac{1}{1-x}}. \tag{1.3}$$

1.7 Problem 7

7. (15 points) Let

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^{2x-x^2}. \tag{1.4}$$

Find a polynomial $P(x)$ such that

$$f(x) - P(x) = o(x^3) \quad \text{as} \quad x \to 0. \tag{1.5}$$

2 Final Wednesday 3 PM

2.1 Problem 1

8. (15 points) Using only the definition of the limit of a sequence, prove that

$$\lim_{n\to\infty} \frac{2n+3}{4n+5} = \frac{1}{2}. \tag{2.1}$$

9. (15 points) Using only the definition of the limit of a sequence, prove that

$$\lim_{n\to\infty} \frac{5n+6}{n+1} = 5. \tag{2.2}$$

2.2 Problem 2

10. (15 points) Prove that the sequence (a_n) given by

$$a_1 = \frac{1}{4}, \quad a_{n+1} = \sqrt{a_n} \tag{2.3}$$

is bounded and monotonic. Compute $\lim a_n$.

11. (15 points) Prove that the sequence (a_n) given by

$$a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt{a_n} \tag{2.4}$$

is bounded and monotonic. Compute $\lim a_n$.

2.3 Problem 3

12. (15 points) Determine if the following series converges

$$\sum_{n=1}^{\infty} (\sqrt{2} - \sqrt[3]{2})(\sqrt{2} - \sqrt[5]{2}) \cdots (\sqrt{2} - \sqrt[2n+1]{2}). \tag{2.5}$$

Justify your answer.

2.4 Problem 4

13. (15 points) Consider the function

$$f(x) = \frac{\log(1-3x)}{x}. \tag{2.6}$$

Note that function f is not defined at $x = 0$.

Construct a *continuous* extension of f defined at $x = 0$ (show that it is indeed continuous at $x = 0$).

14. (15 points) Consider the function

$$f(x) = \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}. \tag{2.7}$$

Note that function f is not defined at $x = 0$.

Construct a *continuous* extension of f defined at $x = 0$ (show that it is indeed continuous at $x = 0$).

2.5 Problem 5

15. (15 points) Let $f : (a, b) \to \mathbb{R}$ satisfy

   (i) f is differentiable on (a, b)

   (ii) f is *unbounded* on (a, b).

   Prove that f', the derivative of f, is also unbounded on (a, b).

   (Hint. You can use proof by contradiction.)

2.6 Problem 6

16. (15 points) Compute the limit

$$\lim_{x\to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}. \tag{2.8}$$

17. (15 points) Compute the limit

$$\lim_{x\to 1} \left(\frac{1}{\log x} - \frac{1}{x-1}\right). \tag{2.9}$$

2.7 Problem 7

18. (15 points) Let

$$f : [-1, +\infty) \to \mathbb{R}, \quad f(x) = \sqrt{1+x}. \tag{2.10}$$

Show that

$$\left| f(x) - \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)\right| \le \frac{1}{16} \tag{2.11}$$

for $x \in [0, 1]$.

(Hint. Use Taylor's formula with remainder in Lagrange's form.)