MATH 142A Introduction to Analysis - FINAL

Winter 2021

March 21, 2021

1 Final Tuesday 8 PM

1.1 Problem 1

1. (15 points) Let (a_n) and (b_n) be two sequences of real numbers such that the sequence $(a_n + b_n)$ is bounded and $\lim a_n = 0$.

   Prove that $\lim a_n b_n = 0$.

Solution. Sequence (a_n) converges, therefore by Theorem 9.1 (a_n) is bounded, which means that there exists $M_1 > 0$ such that

$$\forall n \in \mathbb{N} \quad (|a_n| \leq M_1). \tag{1.1}$$

Since $(a_n + b_n)$ is bounded, there exists $M_2 > 0$ such that

$$\forall n \in \mathbb{N} \quad (|a_n + b_n| \leq M_2). \tag{1.2}$$

We conclude, using the triangle inequality, that for all $n \in \mathbb{N}$

$$|b_n| = |a_n + b_n - a_n| \leq |a_n + b_n| + |a_n| \leq M_1 + M_2, \tag{1.3}$$

the sequence (b_n) is bounded. Now we have that for all $n \in \mathbb{N}$

$$0 \leq |a_n b_n| \leq |a_n|(M_1 + M_2). \tag{1.4}$$

Sequence (a_n) converges to zero, so by Theorem 9.2

$$\lim |a_n|(M_1 + M_2) = 0, \tag{1.5}$$

and (1.4), (1.5) and the Squeeze Lemma (20.14) yield

$$\lim a_n b_n = 0. \tag{1.6}$$

1.2 Problem 2

2. (15 points) Let (a_n) be a Cauchy sequence. Prove that the sequence $\sqrt{a_n}$ is also a Cauchy sequence.

Solution. We may assume that $a_n \geq 0$ to make sure that $\sqrt{a_n}$ is well defined.

Solution 1. Fix $\varepsilon > 0$. By Theorem 10.11, (a_n) converges. Denote by $a \geq 0$ the limit of (a_n), $\lim a_n = a$.

- Case 1: If $a = 0$, then there exists $N_1 \in \mathbb{N}$ such that

  $$n > N \quad \Rightarrow \quad a_n < \frac{\varepsilon^2}{4}, \tag{1.7}$$

  so for any $m, n > N$

  $$|\sqrt{a_n} - \sqrt{a_m}| \leq \sqrt{a_n} + \sqrt{a_m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{1.8}$$

- Case 2: If $a > 0$, then by Theorem 9.11 there exists $N_1 \in \mathbb{N}$ such that

$$n > N_1 \quad \Rightarrow \quad a_n > \frac{a}{4}. \tag{1.9}$$

  (a_n) is a Cauchy sequence, therefore there exists N_2 such that

$$n, m > N_2 \quad \Rightarrow \quad |a_n - a_m| < \sqrt{a}\varepsilon. \tag{1.10}$$

  Then for any $m, n > N := \max\{N_1, N_2\}$

$$|\sqrt{a_n} - \sqrt{a_m}| = \frac{|a_n - a_m|}{\sqrt{a_m} + \sqrt{a_m}} \leq \frac{|a_n - a_m|}{\sqrt{a}} < \varepsilon. \tag{1.11}$$

It follows from (1.8) and (1.11) that there exists $N \in \mathbb{N}$ such that for all $m, n > N$

$$|\sqrt{a_n} - \sqrt{a_m}| < \varepsilon, \tag{1.12}$$

$(\sqrt{a_n})$ is a Cauchy sequence.

Solution 2. Sequence (a_n) is a Cauchy sequence, so by Theorem 10.10 (a_n) is bounded, and there exists $M > 0$ such that for all $n \in \mathbb{N}$

$$a_n \leq M. \tag{1.13}$$

We proved in Lecture 13 that the function $f(x) = \sqrt{x}$ is continuous on $[0, +\infty)$. By Theorem 19.2 (Cantor-Heine Theorem), $f(x)$ is uniformly continuous on $[0, M]$.

Sequence (a_n) is a Cauchy sequence in $[0, M]$, and f is uniformly continuous on $[0, M]$, therefore by Theorem 19.4 the sequence $(f(a_n)) = (\sqrt{a_n})$ is a Cauchy sequence.

Solution 3. Notice that for any $x, y \in [0, +\infty)$, $x < y$ we have

$$y \leq y - x + 2\sqrt{y - x}\sqrt{x} + \sqrt{x} = (\sqrt{y - x} + \sqrt{x})^2, \tag{1.14}$$

so by taking the square root on both sides of the inequality we get

$$\sqrt{y} \leq \sqrt{y - x} + \sqrt{x} \quad \Rightarrow \quad \sqrt{y} - \sqrt{x} \leq \sqrt{y - x}. \tag{1.15}$$

Fix $\varepsilon > 0$. Since (a_n) is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that for all $m, n > N$

$$|a_n - a_m| < \varepsilon^2. \tag{1.16}$$

Then for all $m, n > N$

$$|\sqrt{a_n} - \sqrt{a_m}| \leq \sqrt{|a_n - a_m|} < \varepsilon, \tag{1.17}$$

where we used (1.15) in the first inequality.

1.3 Problem 3

3. (15 points) Determine if the following series converges

$$\sum_{n=1}^{\infty} \frac{n^3\big(\sqrt{2}+(-1)^n\big)^n}{3^n}. \tag{1.18}$$

Justify your answer.

Solution. Use the root test

$$\limsup \sqrt[n]{\frac{n^3\big(\sqrt{2}+(-1)^n\big)^n}{3^n}} = \limsup \frac{\sqrt[n]{n^3}\big(\sqrt{2}+(-1)^n\big)}{3} = \frac{\sqrt{2}+1}{3} < 1, \tag{1.19}$$

where we used that $\sqrt{2} < 2$ and

$$\lim \sqrt[n]{n^3} = 1 \tag{1.20}$$

by the Important Example 3.

It follows from the root test (Theorem 14.9) that the series (1.18) is absolutely convergent.

1.4 Problem 4

4. (15 points) Let function $f : (a, b) \to \mathbb{R}$ be such that

(i) f is bounded on (a, b);

(ii) f is continuous on (a, b);

(iii) f is monotonic on (a, b).

Prove that f is uniformly continuous on (a, b).

(Hint. You can use Theorem 19.5.)

Solution. Consider the sequences (a_n) and (b_n) with

$$a_n = a + \frac{1}{n}, \quad b_n = b - \frac{1}{n}. \tag{1.21}$$

Then the sequences $(f(a_n))$ and $(f(b_n))$ are monotonic and bounded, therefore by Theorem 10.2 $(f(a_n))$ and $(f(b_n))$ converge. Denote

$$A := \lim f(a_n), \quad B := \lim f(b_n), \tag{1.22}$$

and let

$$\tilde{f} : [a, b] \to \mathbb{R}, \quad \tilde{f}(x) = \begin{cases} f(x), & x \in (a, b), \\ A, & x = a, \\ B, & x = b. \end{cases} \tag{1.23}$$

By Theorem 19.5 it is enough to show that $\tilde{f}$ is continuous on $[a, b]$.

Suppose that f is increasing. Fix $\varepsilon > 0$. Then

$$\lim f(a_n) = A \quad \Rightarrow \quad \exists\, N_1(\varepsilon) \in \mathbb{N} \quad \Big(n > N_1(\varepsilon) \;\Rightarrow\; f(a_n) - A < \varepsilon\Big). \tag{1.24}$$

Then for any $x \in (a, a + \frac{1}{N_1(\varepsilon)+1})$ by monotonicity

$$f(x) - A \leq f(a_{N_1(\varepsilon)+1}) - A < \varepsilon, \tag{1.25}$$

and thus $\lim_{x \to a^+} \tilde{f}(x) = A$, $\tilde{f}$ is continuous at a.
Similarly,

$$\lim f(b_n) = b \quad \Rightarrow \quad \exists\, N_2(\varepsilon) \in \mathbb{N} \quad \Big(n > N_2(\varepsilon) \;\Rightarrow\; B - f(b_n) < \varepsilon\Big), \tag{1.26}$$

and for any $x \in (b - \frac{1}{N_2(\varepsilon)+1}, b)$ by monotonicity

$$B - f(x) \leq B - f(b_{N_2(\varepsilon)+1}) < \varepsilon. \tag{1.27}$$

We conclude that $\tilde{f}$ is continuous on $[a, b]$.

If f is decreasing on (a, b), the proof follows from the same argument by switching the roles of A, (a_n) and B, (b_n) in (1.25) - (1.26).

1.5 Problem 5

5. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and satisfy

$$f'(x) = \lambda f(x) \tag{1.28}$$

for some $\lambda > 0$.

Prove that $f(x) = Ce^{\lambda x}$ for some $C \in \mathbb{R}$.

(Hint. Consider function $g(x) = f(x)e^{-\lambda x}$ and its derivative.)

Solution. Consider $g(x) = f(x)e^{-\lambda x}$. Then using the product rule and (1.28) we get

$$g'(x) = f'(x)e^{-\lambda x} - f(x)\lambda e^{-\lambda x} = \lambda f(x)e^{-\lambda x} - f(x)\lambda e^{-\lambda x} = 0. \tag{1.29}$$

Therefore, $g \in D(\mathbb{R})$ and $g'(x) = 0$ for all $x \in \mathbb{R}$. By Corollary 29.4, there exists $C \in \mathbb{R}$ such that

$$g(x) = f(x)e^{-\lambda x} = C. \tag{1.30}$$

We conclude that $f(x) = Ce^{\lambda x}$.

1.6 Problem 6

6. (15 points) Compute the limit

$$\lim_{x\to 1} x^{\frac{1}{1-x}}. \tag{1.31}$$

Solution. First, write

$$x^{\frac{1}{1-x}} = e^{\log x^{\frac{1}{1-x}}} = e^{\frac{1}{1-x}\log x}. \tag{1.32}$$

By the L'Hôpital's rule,

$$\lim_{x\to 1} \frac{\log x}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1. \tag{1.33}$$

Therefore, by the continuity of $x \mapsto e^x$, we get that

$$\lim_{x\to 1} e^{\frac{\log x}{1-x}} = e^{\lim_{x\to 1} \frac{\log x}{1-x}} = e^{-1}. \tag{1.34}$$

1.7 Problem 7

7. (15 points) Let

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^{2x-x^2}. \tag{1.35}$$

Find a polynomial $P(x)$ such that

$$f(x) - P(x) = o(x^3) \quad \text{as} \quad x \to 0. \tag{1.36}$$

Solution. Compute the derivatives of f

$$f'(x) = e^{2x-x^2}(2-2x), \tag{1.37}$$

$$f''(x) = e^{2x-x^2}(2-2x)^2 - 2e^{2x-x^2} = e^{2x-x^2}((2-2x)^2 - 2), \tag{1.38}$$

$$f'''(x) = e^{2x-x^2}((2-2x)^2 - 2)(2-2x) + e^{2x-x^2}(-4(2-2x)). \tag{1.39}$$

We see that $f \in D^{(3)}(\mathbb{R})$. By applying the local Taylor's theorem with the remainder in Peano's form we have

$$f(x) - (f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3) = o(x^3) \quad \text{as} \quad x \to 0. \tag{1.40}$$

Therefore

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + 2x + \frac{2}{2}x^2 - \frac{4}{6}x^3 = 1 + 2x + x^2 - \frac{2}{3}x^3. \tag{1.41}$$

2 Final Wednesday 3 PM

2.1 Problem 1

8. (15 points) Using only the definition of the limit of a sequence, prove that

$$\lim_{n\to\infty} \frac{2n+3}{4n+5} = \frac{1}{2}. \tag{2.1}$$

Solution. Fix $\varepsilon > 0$. For any $n \in \mathbb{N}$ we have that

$$\Big|\frac{2n+3}{4n+5} - \frac{1}{2}\Big| = \Big|\frac{4n+6-(4n+5)}{2(4n+5)}\Big| = \frac{1}{8n+10} < \frac{1}{8n}. \tag{2.2}$$

Therefore, for any $n > [\frac{1}{8\varepsilon}]$ we get

$$\Big|\frac{2n+3}{4n+5} - \frac{1}{2}\Big| < \frac{1}{8n} < \frac{8\varepsilon}{8} = \varepsilon. \tag{2.3}$$

By Definition 7.1 $\lim_{n\to\infty} \frac{2n+3}{4n+5} = \frac{1}{2}$.

9. (15 points) Using only the definition of the limit of a sequence, prove that

$$\lim_{n\to\infty} \frac{5n+6}{n+1} = 5. \tag{2.4}$$

Solution. Fix $\varepsilon > 0$. For any $n \in \mathbb{N}$ we have that

$$\Big|\frac{5n+6}{n+1} - 5\Big| = \Big|\frac{5n+6-5(n+1)}{n+1}\Big| = \frac{1}{n+1} < \frac{1}{n}. \tag{2.5}$$

Therefore, for any $n > [\frac{1}{\varepsilon}]$ we get

$$\Big|\frac{5n+6}{n+1} - 5\Big| < \frac{1}{n} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon. \tag{2.6}$$

By Definition 7.1 $\lim_{n\to\infty} \frac{5n+6}{n+1} = 5$.

2.2 Problem 2

10. (15 points) Prove that the sequence (a_n) given by

$$a_1 = \frac{1}{4}, \quad a_{n+1} = \sqrt{a_n} \tag{2.7}$$

is bounded and monotonic. Compute $\lim a_n$.

Solution. First we show that (a_n) is bounded. Indeed, $a_1 < 1$, and for any $n \in \mathbb{N}$

$$a_n < 1 \quad \Rightarrow \quad a_{n+1} = \sqrt{a_n} < 1. \tag{2.8}$$

By the principle of mathematical induction, for all $n \in \mathbb{N}$

$$a_n < 1. \tag{2.9}$$

Similarly, for all $n \in \mathbb{N}$ we have that $a_n > 0$, and we conclude that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$.
Next, for any $n \in \mathbb{N}$

$$a_{n+1} - a_n = \sqrt{a_n} - a_n = \sqrt{a_n}(1 - \sqrt{a_n}) > 0, \tag{2.10}$$

where we used that $a_n \in (0, 1)$. We conclude that (a_n) in increasing.
By Theorem 10.2, sequence (a_n) converges. Denote $a := \lim a_n$. We have that for any $n \in \mathbb{N}$

$$a_{n+1}^2 = a_n. \tag{2.11}$$

If we take the limit on both sides of the equality (2.18), by Theorem 9.4 we get that

$$a^2 = a \quad \Rightarrow \quad a \in \{0, 1\}. \tag{2.12}$$

Since (a_n) is increasing, $a_n \geq \frac{1}{4}$ for all $n \in \mathbb{N}$, and by the corollary to Theorem 9.11 and (2.19) we have that

$$a \geq \frac{1}{4} \quad \Rightarrow \quad a = 1. \tag{2.13}$$

Therefore, $\lim a_n = 1$.

11. (15 points) Prove that the sequence (a_n) given by

$$a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt{a_n} \tag{2.14}$$

is bounded and monotonic. Compute $\lim a_n$.

Solution. (The same argument as in the previous problem). First we show that (a_n) is bounded. Indeed, $a_1 < 1$, and for any $n \in \mathbb{N}$

$$a_n < 1 \quad \Rightarrow \quad a_{n+1} = \sqrt{a_n} < 1. \tag{2.15}$$

By the principle of mathematical induction, for all $n \in \mathbb{N}$

$$a_n < 1. \tag{2.16}$$

Similarly, for all $n \in \mathbb{N}$ we have that $a_n > 0$, and we conclude that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$.
Next, for any $n \in \mathbb{N}$

$$a_{n+1} - a_n = \sqrt{a_n} - a_n = \sqrt{a_n}(1 - \sqrt{a_n}) > 0, \tag{2.17}$$

where we used that $a_n \in (0, 1)$. We conclude that (a_n) in increasing.
By Theorem 10.2, sequence (a_n) converges. Denote $a := \lim a_n$. We have that for any $n \in \mathbb{N}$

$$a_{n+1}^2 = a_n. \tag{2.18}$$

If we take the limit on both sides of the equality (2.18), by Theorem 9.4 we get that

$$a^2 = a \quad \Rightarrow \quad a \in \{0, 1\}. \tag{2.19}$$

Since (a_n) is increasing, $a_n \geq \frac{1}{3}$ for all $n \in \mathbb{N}$, and by the corollary to Theorem 9.11 and (2.19) we have that

$$a \geq \frac{1}{3} \quad \Rightarrow \quad a = 1. \tag{2.20}$$

Therefore, $\lim a_n = 1$.

2.3 Problem 3

12. (15 points) Determine if the following series converges

$$\sum_{n=1}^{\infty} (\sqrt{2} - \sqrt[3]{2})(\sqrt{2} - \sqrt[5]{2}) \cdots (\sqrt{2} - \sqrt[2n+1]{2}). \tag{2.21}$$

Justify your answer.

Solution. Use the ratio test. Denote the n-th term of the series by a_n

$$a_n := (\sqrt{2} - \sqrt[3]{2})(\sqrt{2} - \sqrt[5]{2}) \cdots (\sqrt{2} - \sqrt[2n+1]{2}). \tag{2.22}$$

Then

$$\lim \frac{a_{n+1}}{a_n} = \lim(\sqrt{2} - \sqrt[2n+3]{2}) = \sqrt{2} - 1, \tag{2.23}$$

where we used that $\lim \sqrt[n]{2} = 1$ (Important Example 4), and that any subsequence of a convergent sequence converges to the same limit (Theorem 11.3).

Since $\sqrt{2} < 2$, we have that

$$\lim \frac{a_{n+1}}{a_n} < 1, \tag{2.24}$$

and thus by the ration test (Theorem 14.8) the series (2.21) converges.

2.4 Problem 4

13. (15 points) Consider the function

$$f(x) = \frac{\log(1 - 3x)}{x}. \tag{2.25}$$

Note that function f is not defined at $x = 0$.

Construct a *continuous* extension of f defined at $x = 0$ (show that it is indeed continuous at $x = 0$).

Solution. Function $x \mapsto \log(1-3x)$ is defined and continuous on the interval $(-\infty, \frac{1}{3})$, and function $x \mapsto \frac{1}{x}$ is defined and continuous on $\mathbb{R} \setminus \{0\}$. Therefore, the domain of definition of f is $(-\infty, 1/3) \setminus \{0\}$.

In order to construct an extension of f continuous at $x = 0$ we introduce the function

$$\tilde{f} : \Big(-\infty, \frac{1}{3}\Big) \to \mathbb{R}, \quad \tilde{f}(x) = \begin{cases} f(x), & x \neq 0, \\ c, & x = 0. \end{cases} \tag{2.26}$$

$\tilde{f}$ is continuous on $(-\infty, 1/3) \setminus \{0\}$, and we have to determine the value c for which $\tilde{f}$ is continuous at zero.

By definition, $\tilde{f}$ is continuous at $x = 0$ if

$$\lim_{x \to 0} \tilde{f}(x) = \tilde{f}(0) = c. \tag{2.27}$$

By using the Important Example 13 and Theorem 20.5 (about the limit of a composition of functions) (one can also use the L'Hôpital's rule) we find c

$$\lim_{x \to 0} \tilde{f}(x) = \lim_{x \to 0} \frac{\log(1-3x)}{x} = -3 \lim_{x \to 0} \frac{\log(1-3x)}{-3x} = -3 \cdot 1 = -3. \tag{2.28}$$

The continuous extension of f is given by (2.26) with $c = -3$.

14. (15 points) Consider the function

$$f(x) = \frac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1}. \tag{2.29}$$

Note that function f is not defined at $x = 0$.

Construct a *continuous* extension of f defined at $x = 0$ (show that it is indeed continuous at $x = 0$).

Solution. Function f is defined and continuous on the interval $[-1, +\infty) \setminus \{0\}$.

In order to construct an extension of f continuous at $x = 0$ we introduce the function

$$\tilde{f} : [-1, \infty) \to \mathbb{R}, \quad \tilde{f}(x) = \begin{cases} f(x), & x \neq 0, \\ c, & x = 0. \end{cases} \tag{2.30}$$

$\tilde{f}$ is continuous on $[-1, \infty) \setminus \{0\}$, and we have to determine the value c for which $\tilde{f}$ is continuous at zero.

By definition, $\tilde{f}$ is continuous at $x = 0$ if

$$\lim_{x \to 0} \tilde{f}(x) = \tilde{f}(0) = c. \tag{2.31}$$

We find c by computing the limit (one can also use the L'Hôpital's rule)

$$\lim_{x \to 0} \tilde{f}(x) = \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1} = \lim_{x \to 0} \frac{1+x-1}{\sqrt{1+x}+1} \cdot \frac{(\sqrt[3]{1+x})^2 + \sqrt[3]{1+x} + 1}{1+x-1} = \frac{3}{2}, \tag{2.32}$$

since

$$(\sqrt{1+x}-1)(\sqrt{1+x}+1) = 1+x-1 = x \tag{2.33}$$

and

$$(\sqrt[3]{1+x}-1)((\sqrt[3]{1+x})^2+\sqrt[3]{1+x}+1) = 1+x-1 = x. \tag{2.34}$$

The continuous extension of f is given by (2.30) with $c = 3/2$.

2.5 Problem 5

15. (15 points) Let $f : (a, b) \to \mathbb{R}$ satisfy

   (i) f is differentiable on (a, b)

   (ii) f is *unbounded* on (a, b).

   Prove that f', the derivative of f, is also unbounded on (a, b).

   (Hint. You can use proof by contradiction.)

Solution. Suppose that f' is bounded on (a, b). This means that there exists $M > 0$ such that for all $x \in (a, b)$

$$|f'(x)| \le M. \tag{2.35}$$

Fix a point $x_0 \in (a, b)$. Then for any $x \in (a, b)$, $x > x_0$, we have that

$$f \in C([x_0, x]), \quad f \in D((x_0, x)). \tag{2.36}$$

It follows from the mean value theorem (Theorem 29.3) applied to the function f on the interval $[x_0, x]$ that there exists $c \in (x_0, x)$ for which

$$f(x) - f(x_0) = f'(c)(x - x_0). \tag{2.37}$$

Therefore, by using (2.33) we get the following bound

$$|f(x)| = |f(x_0) + f'(c)(x - x_0)| \le |f(x_0)| + |f'(c)||x - x_0| \le |f(x_0)| + M|b - a|, \tag{2.38}$$

where we used that $c \in (a, b)$ and $|x - x_0| < |b - a|$.

Similarly, for any $x \in (a, b)$, $x < x_0$, by applying the mean value theorem to f on $[x, x_0]$ we get

$$f(x_0) - f(x) = f'(c)(x_0 - x), \tag{2.39}$$

which again leads to the bound

$$|f(x)| \le |f(x_0)| + |f'(c)||x - x_0| \le |f(x_0)| + M|b - a|. \tag{2.40}$$

We conclude that if f' is bounded on (a, b), then the function f is bounded on (a, b) by $|f(x_0)| + M|b - a|$, which contradicts to the assumption that f is unbounded on (a, b). The derivative f' is thus unbounded on (a, b).

2.6 Problem 6

16. (15 points) Compute the limit

$$\lim_{x \to 0} \frac{x(e^x+1) - 2(e^x-1)}{x^3}. \tag{2.41}$$

Solution. Both numerator and denominator tend to zero, so by applying the L'Hôpital's rule (twice) we get

$$\lim_{x \to 0} \frac{x(e^x+1) - 2(e^x-1)}{x^3} = \lim_{x \to 0} \frac{(e^x+1) + xe^x - 2e^x}{3x^2} = \lim_{x \to 0} \frac{1 + xe^x - e^x}{3x^2} \tag{2.42}$$

$$= \lim_{x \to 0} \frac{e^x + xe^x - e^x}{6x} = \lim_{x \to 0} \frac{e^x}{6} = \frac{1}{6}. \tag{2.43}$$

17. (15 points) Compute the limit

$$\lim_{x \to 1} \Big(\frac{1}{\log x} - \frac{1}{x-1} \Big). \tag{2.44}$$

Solution. First rewrite the above function as

$$\frac{1}{\log x} - \frac{1}{x-1} = \frac{x - 1 - \log x}{\log x (x-1)}. \tag{2.45}$$

We see that as x tends to 1, both numerator and denominator tend to zero, so by applying the L'Hôpital's rule (twice) we get

$$\lim_{x \to 1} \frac{x - 1 - \log x}{\log x (x-1)} = \lim_{x \to 1} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \log x} \tag{2.46}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{2}. \tag{2.47}$$

2.7 Problem 7

18. (15 points) Let

$$f : [-1, +\infty) \to \mathbb{R}, \quad f(x) = \sqrt{1+x}. \tag{2.48}$$

Show that

$$\Big| f(x) - \Big(1 + \frac{x}{2} - \frac{x^2}{8}\Big) \Big| \le \frac{1}{16} \tag{2.49}$$

for $x \in [0, 1]$.

(Hint. Use Taylor's formula with remainder in Lagrange's form.)

Solution. Compute the derivatives of f

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \tag{2.50}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}, \tag{2.51}$$

$$f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2}. \tag{2.52}$$

We see that $f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$, and thus

$$1 + \frac{x}{2} - \frac{x^2}{8} \tag{2.53}$$

coincides with the Taylor's polynomial of order 2 of f at $x = 0$. Therefore, by the Taylor's theorem with the remainder in Lagrange's from (Corollary 31.3), for any $x \in (0, 1]$ there exists a number ξ between 0 and x such that

$$f(x) - \left(1 + \frac{x}{2} - \frac{x^2}{8}\right) = \frac{f^{(3)}(\xi)}{3!}x^3. \tag{2.54}$$

Plugging in the expression of $f^{(3)}$ computed earlier we get the following bound

$$\left|f(x) - \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)\right| = \frac{\frac{3}{8}(1+\xi)^{-5/2}}{3!}x^3 \leq \frac{3}{8 \cdot 3!} = \frac{1}{16}, \tag{2.55}$$

where we used that for $x \in (0, 1]$ and $\xi \in (0, 1]$

$$x^3 \leq 1, \quad \text{and} \quad \frac{1}{(1+\xi)^{5/2}} \leq 1. \tag{2.56}$$