MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Birth and death processes.

Next: PK 6.5

Week 2:

- homework 1 (due Friday April ~~7~~8)
- NO IN PERSON LECTURE ON WEDNESDAY

The Yule process

Setting: In a certain population each individual during any (small) time interval of length h gives a birth to one new individual with probability $\beta h + o(h)$, independently of other members of the population. All members of the population live forever. At time 0 the population consists of one individual.

Question: What is the distribution on the size of the population at a given time t?

The Yule process

Let X_t, $t \geq 0$, be the size of the population at time t.

$X_0 = 1$ (start from one common ancestor).

Compute $\tilde{P}_n(t) = P(X_t = n \mid X_0 = 1)$

If $X_t = n$, then during a time interval of length h

(a) $P(X_{t+h} = n+1 \mid X_t = n) = n\beta h + o(h)$

(b) $P(X_{t+h} = n \mid X_t = n) = 1 - n\beta h + o(h)$ $\qquad h \to 0$

(c) $P(X_{t+h} > n+1 \mid X_t = n) = o(h)$

all n indiv. give 0 births

(b) $P(0 \text{ births} \mid X_t = n) = (1 - \beta h + o(h))^n = 1 - n\beta h + o(h)$

(a), (b), (c) $\Rightarrow (X_t)_{t \geq 0}$ is a pure birth process with rates $\lambda_n = n\beta$

$\{\tilde{P}_n(t)\}$ satisfies the system of differential equations

The Yule process

$$
(*)\quad
\begin{cases}
\tilde{P}_{10}'(t) = -\beta \tilde{P}_{10}(t) & \tilde{P}_{10}(0) = 1 \\
\tilde{P}_{21}'(t) = -2\beta \tilde{P}_{21}(t) + \beta \tilde{P}_{10}(t) & P_{21}(0) = 0 \\
\vdots & \vdots \\
\tilde{P}_{n\,n-1}'(t) = -n\beta \tilde{P}_{n\,n-1}(t) + (n-1)\beta \tilde{P}_{n-1\,n-2}(t) & P_{n\,n-1}(0) = 0 \\
\vdots & \vdots
\end{cases}
$$

The same system with shifted indices

$$\tilde{P}_1(t) = P_0(t) \qquad \tilde{P}_n(t) = P_{n-1}(t) \qquad \text{with } \lambda_n = \beta(n+1)$$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right) \qquad \lambda_0 \cdots \lambda_{n-1} = \beta^n n!$$

$$B_{kn} = \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n} \frac{1}{\lambda_\ell - \lambda_k} \qquad B_{kn} = \prod_{\ell=0}^{k-1} \frac{1}{\lambda_\ell - \lambda_k} \prod_{\ell=k+1}^{n} \frac{1}{\lambda_\ell - \lambda_k} = \frac{1}{\beta^n (-1)^k k! \, (n-k)!}$$

The Yule process

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

$$= \sum_{k=0}^{n} \cancel{\beta^n}\, n! \, \frac{(-1)^k}{\cancel{\beta^n}\, k!\, (n-k)!} e^{-\beta(k+1)t}$$

$$= e^{-\beta t} \sum_{k=0}^{n} \binom{n}{k} \left(-e^{-\beta t}\right)^k 1^{n-k}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$= e^{-\beta t} \left(1 - e^{-\beta t}\right)^n$$

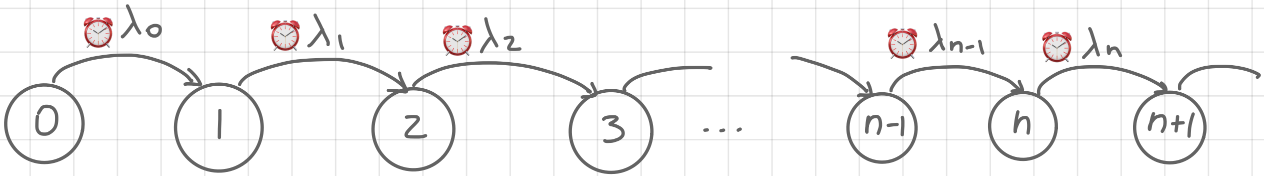
$$\tilde{P}_n(t) = P_{n-1}(t) = e^{-\beta t}\left(1 - e^{-\beta t}\right)^{n-1}$$

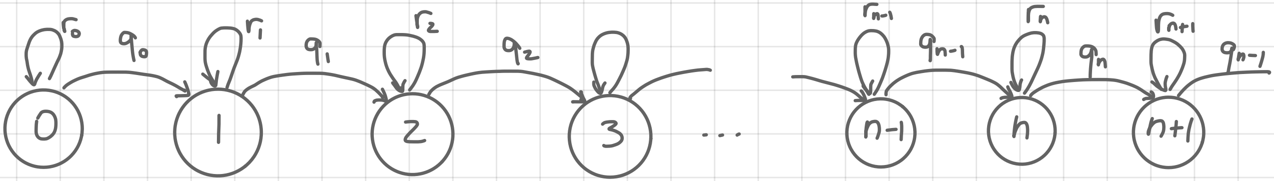
$$q = e^{-\beta t}$$

$$\hookrightarrow \quad P(X_t = n) = q(1-q)^{n-1}$$

$$\hookrightarrow \quad X_t \sim \mathrm{sGeom}\left(e^{-\beta t}\right)$$

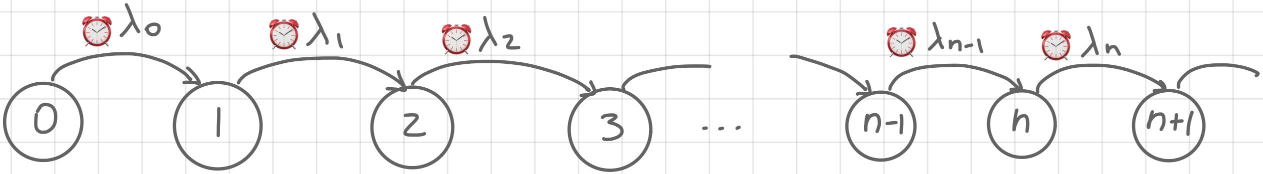
Graphical representation. Exponential sojourn times



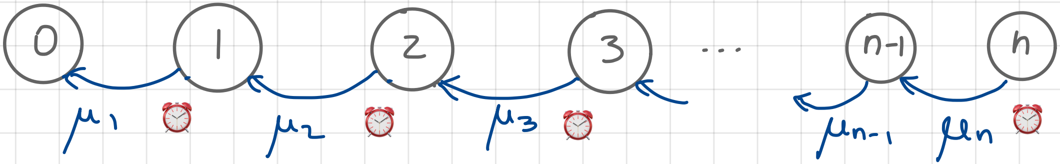


Pure death processes

Pure birth process



What if the chain moves in the opposite direction?



Pure death process:

- exponential sojourn times with rates μ_i
- only negative jumps of magnitude 1 allowed

Pure death processes

Infinitesimal description:

Pure death process $(X_t)_{t\geq 0}$ of rates $(\mu_k)_{k=1}^{N}$ is a continuous time MC taking values in $\{0, 1, 2, --, N-1, N\}$ (state 0 is absorbing) with stationary infinitesimal transition probability functions

(a) $P_{k,k-1}(h) = \mu_k h + o(h)$ $k=1, --, N$

$h \to 0$

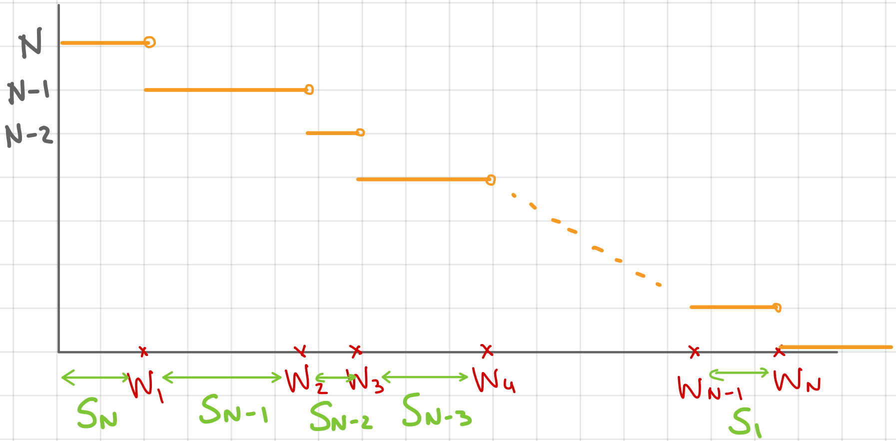
(b) $P_{kk}(h) = 1 - \mu_k h + o(h)$, $k=1, ..., N$

(c) $P_{kj}(h) = 0$ for $j > k$.

State 0 is absorbing $(\mu_0 = 0)$

Pure death process

$S_k \sim \operatorname{Exp}(\mu_k)$



Sojourn time / jump description:

Pure death process of rates $(\mu_k)_{k=1}^N$ is a nonincreasing right-continuous process taking values in $\{0, 1, \dots, N\}$

- with sojourn times $S_1, S_2, S_3, \dots, S_N$ being independent exponential r.v.s of rates $\mu_1, \mu_2, \dots, \mu_N$ and
- jumps $X_{W_{i+1}} - X_{W_i} = -1$ of magnitude 1

Differential equations for pure death processes

Define $P_n(t) = P(X_t = n \mid X_0 = N)$ distribution of X_t

$\uparrow$ starting in state N

(a), (b), (c) implies (check)

$$\begin{cases} P_n'(t) = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t) & \text{for } n = 0 \dots N-1 \\ & (\text{note that } \mu_0 = 0) \\ P_N'(t) = -\mu_N P_N(t) & \end{cases}$$

Initial conditions: $P_N(0) = 1$, $P_n(0) = 0$ for $n = 0, \dots, N-1$

Solve recursively: $P_N(t) = e^{-\mu_N t} \to P_{N-1}(t) \to \dots \to P_0(t)$

General solution (assume $\mu_i \neq \mu_j$)

$$P_n(t) = \mu_{n+1} \cdots \mu_N \left(A_{nn} e^{-\mu_n t} + \dots + A_{N,n} e^{-\mu_N t} \right), \quad A_{kn} = \prod_{\substack{l=n \\ l \neq k}}^{N} \frac{1}{\mu_l - \mu_k}$$

Linear death process [Discussion section]

Similar to Yule process:

death rate is proportional to the size of the population

$\mu_k = \alpha k$ (linear dependence on k)

Compute $P_n(t)$: • $\mu_{n+1} \cdots \mu_N = \alpha^{N-n} \frac{N!}{n!}$

• $A_{kn} = \prod_{\substack{\ell=n \\ \ell \neq k}}^{N} \frac{1}{\mu_\ell - \mu_k} = \frac{1}{\alpha^{N-n} (-1)^{n-k} (k-n)! (N-k)!}$ $\{ \mu_\ell - \mu_k = \alpha(\ell - k)$

• $P_n(t) = \alpha^{N-n} \frac{N!}{n!} \cdot \frac{1}{\alpha^{N-n}} \sum_{k=n}^{N} \frac{1}{(-1)^{n-k} (k-n)! (N-k)!} \cdot e^{-k\alpha t}$ $\begin{cases} j = k-n \\ k = j+n \end{cases}$

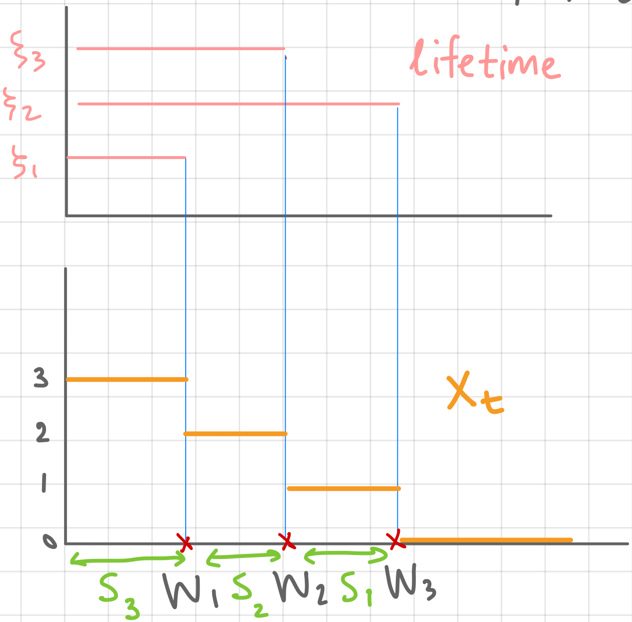
$= \frac{N!}{n!} \sum_{j=0}^{N-n} \frac{(-1)^j e^{-(j+n)\alpha t}}{j! (N-n-j)!}$

$= \frac{N!}{n!} e^{-n\alpha t} \sum_{j=0}^{N-n} \frac{1}{j! (N-n-j)!} \left(-e^{-\alpha t}\right)^j = \frac{N!}{n! (N-n)!} e^{-n\alpha t} \left(1 - e^{-\alpha t}\right)^{N-n}$

$X_t \sim \text{Bin}(N, e^{-\alpha t})$

Interpretation of $X_t \sim Bin(n, e^{-\alpha t})$ [Discussion section]

Consider the following process: Let ξ_i, $i = 1 \dots N$, be i.i.d. r.v.s, $\xi_i \sim Exp(\alpha)$. Denote by X_t the number of ξ_i's that are bigger than t (ξ_i is the lifetime of an individual, X_t = size of the population at t). $X_0 = N$.



Then: $S_k \sim Exp(\alpha k)$, independent

↳ $(X_t)_{t \geq 0}$ is a pure death process

Probability that an individual survives to time t is $e^{-\alpha t}$

Probability that exactly n individuals survive to time t is

$$\binom{N}{n} e^{-\alpha t n} (1 - e^{-\alpha t})^{N-n} = P(X_t = n)$$