MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Poisson process as a renewal process. Other examples

Next: PK 7.4-7.5, Durrett 3.1

Week 6:

- homework 5 (due Friday, May 6)
- regrades for Midterm 1 active until May 7, 11PM

Renewal density

$f * f(t) = \int f(t-x) f(x) dx$

<u>Proposition</u> Let $N(t)$ be a renewal process with continuous interrenewal times X_i having density $f(x)$. Denote $m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$. Then $M(t) = \int_0^t m(x) dx$

and $m(t) = f(t) + m * f(t)$ (*)

renewal density

<u>Proof</u>: $\frac{d}{dt} F^{*n}(t) = \left(\frac{d}{dt} F^{*(n-1)}\right) * f(t) = f^{*n}(t)$ ∎

<u>Example</u>: Compute the renewal density for PP using (*).

$f(x) = \lambda e^{-\lambda x}$, so (*) becomes

$$m(t) = \lambda e^{-\lambda t} + \int_0^t m(t-x) \lambda e^{-\lambda x} dx = \lambda e^{-\lambda t} + \int_0^t m(x) \lambda e^{-\lambda(t-x)} dx$$

$$= \lambda e^{-\lambda t}\left(1 + \int_0^t m(x) e^{\lambda x} dx\right)$$

(cont.)

$$e^{\lambda t} m(t) = \lambda \left(1 + \int_0^t m(x) e^{\lambda x}\, dx\right) \leftarrow \text{differentiate}$$

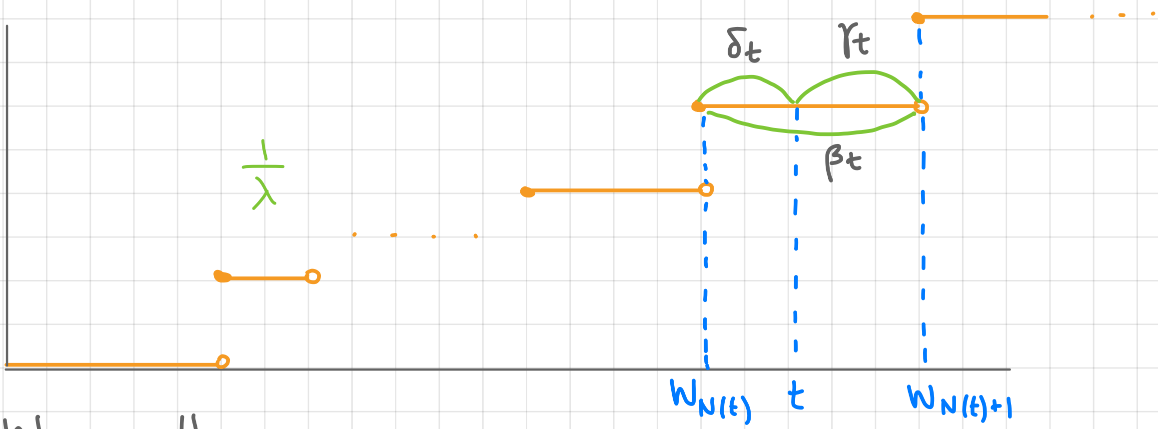
$$\begin{cases} \dfrac{d}{dt}\left(e^{\lambda t} m(t)\right) = \lambda\left(m(t) e^{\lambda t}\right) \\ m(0) = \lambda \end{cases} \Rightarrow \quad e^{\lambda t} m(t) = \lambda e^{\lambda t}$$

$$m(t) = \lambda$$

Indeed, $\quad M(t) = \displaystyle\int_0^t m(x)\,dx = \int_0^t \lambda\, dx = \lambda t$

Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks

1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

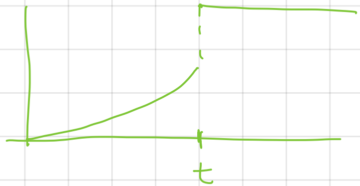
Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life $\gamma_t \sim \text{Exp}(\lambda)$

$$P(\gamma_t > x) = P\big(N(t+x) - N(t) = 0\big) = P\big(N(x) = 0\big) = e^{-\lambda x}$$

- current life δ_t



$$P(\delta_t > x) = \begin{cases} 0, & x \geq t \\ P\big(N(t-x) = N(t)\big) = P\big(N(t) - N(t-x) = 0\big) = e^{-\lambda x}, & x < t \end{cases}$$

- total life $\beta_t = \gamma_t + \delta_t$

$$E(\gamma_t + \delta_t) = \frac{1}{\lambda} + E(\delta_t) = \frac{1}{\lambda} + \int_0^\infty P(\delta_t > x)\,dx$$

$$= \frac{1}{\lambda} + \int_0^t e^{-\lambda x}\,dx = \frac{1}{\lambda} + \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right) \longrightarrow \frac{2}{\lambda} \quad \text{as } t \to \infty$$

Excess life and current life of PP (cont.)

- Joint distribution of (γ_t, δ_t)

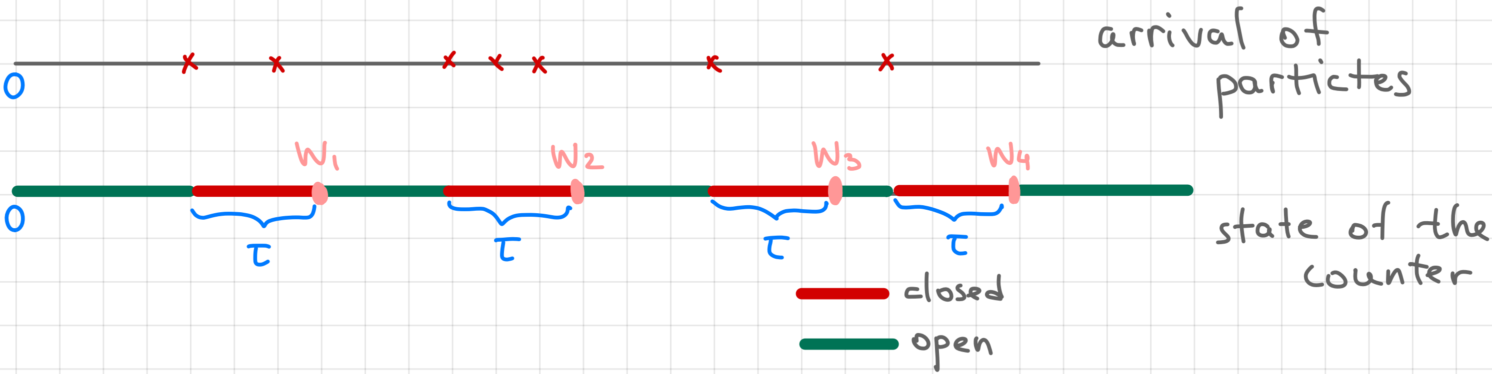
$$P(\gamma_t > x, \delta_t > y) = \begin{cases} 0, & \text{if } y \geq t \\ P(N(t-y) = N(t+x)) = e^{-\lambda(x+y)}, & y < t \end{cases}$$

$\Rightarrow$ γ_t and δ_t are independent (for PP)

Other renewal processes

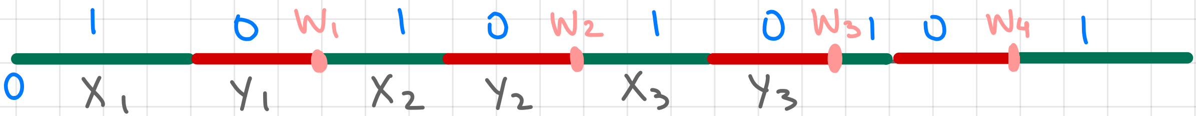
- traffic flow: distances between successive cars are assumed to be i.i.d. random variables

- counter process: particles/signals arrive on a device and lock it for time τ; particles arrive according to a PP; times at which the counter unlocks form a renewal process



Other renewal processes

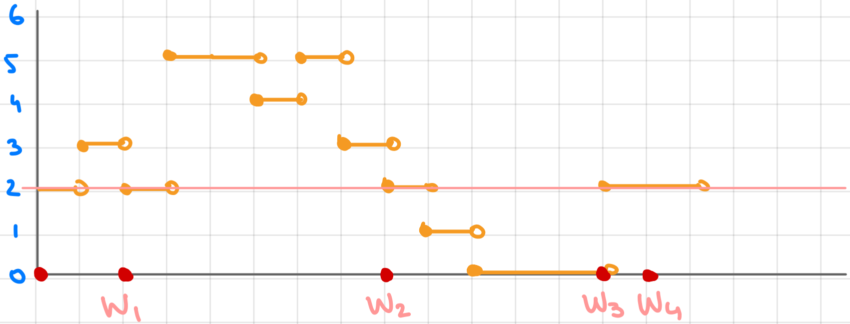
- more generally, if a component has two states (0/1, operating/non-operating etc), switches between them, times spent in 0 are X_i, times spent in 1 are Y_i, $(X_i)_{i=1}^{\infty}$ i.i.d., $(Y_i)_{i=1}^{\infty}$ i.i.d., then the times of switching from 0 to 1 form a renewal process with interrenewal times $X_i + Y_i$



Other renewal processes

- Markov chains: if $(Y_n)_{n\geq 0}$, $Y_n \in \{0,1,\dots\}$ is a recurrent MC starting from $Y_0 = k$, then the times of returns to state k form a renewal process. More precisely define

$$W_1 = \min\{n>0 : Y_n = k\}$$
$$W_p = \min\{n > W_{p-1} : Y_n = k\}$$



Example with $k=2$

Similarly for continuous time MCs.

Strong Markov property!

Other renewal processes

- Queues. Consider a single-server queueing process

(i) if customer arrival times form a renewal process then the times of the starts of successive idle periods generate a second renewal time

(ii) if customes arrive according to a Poisson process, then the times when the server passes from busy to free form a renewal process