MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 2.5, Durrett 5.1-5.2

Week 7:

- homework 6 (due Monday, May 16, week 8)

Midterm 2: Wednesday, May 18

Example: Age replacement policies (PK, p. 363)

X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures

$N(t) = \#$ replacements on $[0,t]$, $Q(t) = \#$ failure replacements on $[0,t]$

Last time:

$$\frac{E(N(t))}{t} \approx \frac{1}{\mu_T} \quad \text{for large } t$$

$Q(t)$ renewal process with interrenewal times Y_i and

$$Y_1 = L \cdot T + Z \quad \text{with} \quad P(L \geq n) = (1 - F(T))^n, \quad P(Z \leq z) = \frac{F(z)}{F(T)}$$

Example: Age replacement policies (PK, p. 363)

Now we can compute the long-run rate of the replacements due to failures

$E(Y_1) =$

$E(L) =$

$E(Z) =$, so

$E(Y_1) =$

Applying the elementary renewal theorem to $Q(t)$

Example: Age replacement policies (PK, p. 363)

Suppose that the cost of one replacement is K, and each replacement due to a failure costs additional c. Then, in the long run the total amount spent on the replacements of the component per unit of time is given by

$$C(T) \approx$$

If we are given c, K and the distribution of the component's lifetime F, we can try to minimize the overall costs by choosing the optimal value of T.

Example: Age replacement policies (PK, p. 363)

For example, if $K=1$, $C=4$ and $X_1 \sim \mathrm{Unif}[0,1]$ ($F(x) = x \mathbb{1}_{[0,1]}$)

For $T \in [0,1]$, $\mu_T =$ and

the average (per unit of time) long-run costs are

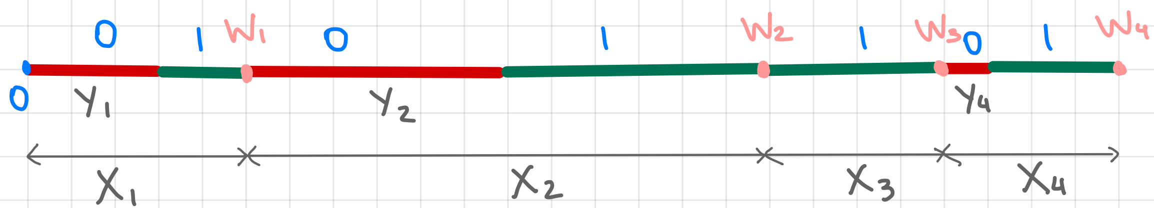
$$C(T) =$$

$$\frac{d}{dT} C(T) =$$

Two component renewals

Consider the following model:

- $(X_i)_{i=1}^{\infty}$ are interrenewal times
- at each moment of time the system $S(t)$ can be in one of two states: $S(t) = 0$ or $S(t) = 1$
- random variables Y_i denote the part of X_i during which the system is in state 0, $0 \le Y_i \le X_i$
- collection $((X_i, Y_i))_{i=1}^{\infty}$ is i.i.d.



Q: In the long run (for large t), what is the probability that the system is in state 1 at time t?

Two component renewals

Thm.

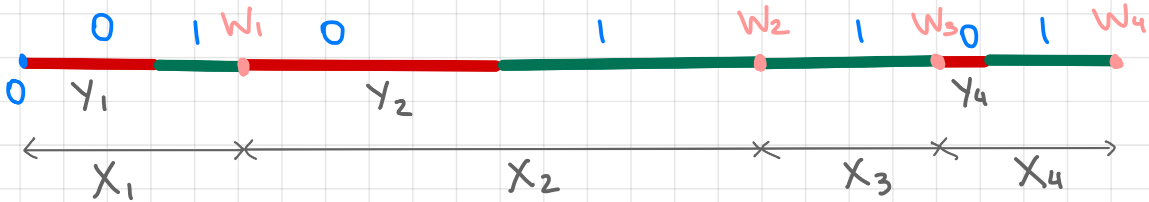
$$\lim_{t\to\infty} P(S(t)=0) =$$

Proof. Denote $g(t) =$. Then

$$g(t) =$$

If $t < x$, then $P(S(t)=0 \mid X_1 = x) =$

If $t \geq x$, then $P(S(t)=0 \mid X_1 = x) =$



Two component renewals

$$g(t) = \int_t^{\infty} \qquad\qquad + \int_0^t$$

Function g satisfies the renewal equation

$$g(t) =$$

Note that $Y_1 \le X_1$, therefore $P(Y_1 > t \mid X_1 = x) =$ for $x < t$,

$$h(t) =$$

$$\int_0^{\infty} h(t)\,dt =$$

From the Key renewal theorem $\lim_{t \to \infty} g(t) =$

Example: the Peter principle

Setting:
- infinite population of candidates for certain position
- fraction p of the candidates are competent, $q = 1-p$ are incompetent
- if a competent person is chosen, after time C_i he/she gets promoted
- if an incompetent person is chosen, he/she remains in the job until retirement (r.v. I_j)
- once the position is open again, the process repeats

Question: What fraction of time, denoted f, is the position held by an incompetent person on average in the long run?

Example: the Peter principle

Denote $X_i = \begin{cases} \quad , & \text{if occupied by a competent person} \\ \quad , & \text{if occupied by an incompetent person} \end{cases}$

$Y_i = \begin{cases} \quad , & \text{if occupied by a competent person} \\ \quad , & \text{if occupied by an incompetent person} \end{cases}$

KRT for two component renewals can be applied to $((X_i, Y_i))_{i=1}^{\infty}$

If $S(t) = 0$ if the person is incompetent, then

$$\lim_{t\to\infty} P(S(t)=0) = \frac{E(Y_1)}{E(X_1)} \qquad \text{and}$$

$$f := \lim_{t\to\infty} \Big(\qquad\qquad \Big) = \lim_{t\to 0} \frac{1}{t} \int_0^t \qquad du =$$

Finally, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$

, then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle (alternative)

Let $X_i = \begin{cases} C_i, & \text{if the i-th person is competent} \\ I_i, & \text{if the i-th person is incompetent} \end{cases}$

$Y_i = \begin{cases} 0, & \text{time occupied by a competent person} \\ I_i, & \text{time occupied by an incompetent person} \end{cases}$

and assume that $|X_i| < K$. Then using

$$\leq E\left(\frac{1}{t}\int_0^t \mathbb{1}_{\{S(u)=0\}}\,du\right) \leq$$

Again, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$

, then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle

If we take , then

$$f =$$