MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Martingales

Next: PK 8.1

Week 9:

- homework 7 (due Friday, May 27)

Maximal inequality for nonnegative martingales

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values.

For any $\lambda > 0$ and $m \in \mathbb{N}$

$$P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n\geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Proof. We prove (1), (2) follows by taking the limit $m\to\infty$.

Take the vector $(X_0, X_1, \dots, X_m)$ and partition the sample space wrt the index of the first r.v. rising above λ

$$1 = \mathbb{1}_{X_0\geq\lambda} + \mathbb{1}_{X_0<\lambda, X_1\geq\lambda} + \dots + \mathbb{1}_{X_0<\lambda,\dots,X_{m-1}<\lambda, X_m\geq\lambda} + \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda}$$

Compute $E(X_m) = E(X_m\cdot 1)$ using the above partition

Proof of the maximal inequality

$$E(X_m) = \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda}) + \underbrace{E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda})}^{\geq 0}$$

$$\geq \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda})$$

Compute $E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda})$ by conditioning on

$X_0, X_1, \dots, X_{n-1}, X_n$:

$E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda})$

$=$

$=$

$=$

Sum for all n

$E(X_m) \geq$

Example

A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gamblers bets fraction p of his current fortune, wins with probability $\frac{1}{2}$, loses with probability $\frac{1}{2}$. Estimate the probability that the gambler ever doubles the initial fortune.

Denote by $Z_n, n \geq 0$, the gambler's fortune after n-th game.

Denote

Then

Martingale transform

In the previous example the stake in n-th game is $p Z_{n-1}$. What if we choose another strategy?

<u>Def</u> Let $(X_n)_{n\geq 0}$ be a nonnegative martingale, and let $(C_n)_{n\geq 0}$ be a stochastic process with $C_n = f_n(X_0, \dots, X_{n-1})$. Then the stochastic process

is called the

Think of
- $X_k - X_{k-1}$ as the winning per unit stake in k-th game
- C_k as your stake in k-th game
  decision is made based on the previous history
- $(C \cdot X)_n$ as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \cdot X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \geq 0}$ is a martingale

Proof: $E(Z_{n+1} \mid Z_0, \dots, Z_n) =$

$=$

Note that

If $Z_n > 0$, then $C_1 > 0, \dots, C_n > 0$,

and

$E(Z_{n+1} \mid Z_0, \dots, Z_n) =$

$=$

If $Z_n = 0$, then $C_{n+1} = 0$ and $E(Z_{n+1} \mid Z_0, \dots, Z_n) = 0 = Z_n$

Gambling example:

Start from the initial fortune $X_0 = 1$. Define

$$Z_n =$$

fortune after n-th game with strategy C

Then $(Z_n)_{n\geq 0}$ is a nonnegative martingale, $E(Z_0)=1$

$$\Rightarrow$$

Convergence of nonnegative martingales

Thm.

If $(X_n)_{n\geq 0}$ is a nonnegative (super)martingale, then with probability 1

and

Example

An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn together with another ball of the same color. Repeat. Denote by X_n the fraction of red ball after n iterations.

Example (cont.)

(i) $(X_n)_{n\geq 0}$ is a martingale

Denote by R_n the number of red balls after n-th iteration

$$R_n =$$

Then

$$E(X_{n+1} \mid X_0, \dots, X_n) =$$

$$=$$

(ii) X_n is nonnegative $\Rightarrow$

(iii) Compute the distribution of X_∞

$$P\left(X_n = \frac{k}{n+2}\right) = \frac{1}{n+1} \quad \text{for } k \in \{1, 2, \dots, n+1\}$$

$$P(X_\infty \leq x) = x, \quad x \in (0,1) \Rightarrow X_\infty \sim \mathrm{Unif}[0,1)$$

Brownian motion

Brownian motion. History

- Critical observation: Robert Brown (1827), botanist, movement of pollen grains in water
- First (?) mathematical analysis of Brownian motion: Louis Bachelier (1900), modeling stock market fluctuations
- Brownian motion in physics: Albert Einstein (1905) and Marian Smoluchowski (1906), explained the phenomenon observed by Brown
- First rigorous construction of mathematical Brownian motion: Norbert Wiener (1923)

$$\text{Brownian motion} \underset{\substack{\uparrow \\ \text{in mathematics}}}{=} \text{Wiener process}$$

Brownian motion. Motivation

- almost all interesting classes of stochastic processes contain Brownian motion: BM is a
  - martingale
  - Markov process
  - Gaussian process
  - Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i)

(ii)

(iii)

$\sigma^2 = 1 \leftarrow$ standard BM

BM as a continuous time continuous space Markov process

Recall: continuous time discrete space MC $(X_t)_{t\geq 0}$ is characterized by the transition probability function

$$P_{ij}(t) =$$

($(X_t)_{t\geq 0}$ has stationary transition probability functions)

In particular, $P(X_{s+t} \in A \mid X_s = i) =$

In the continuous state space case the transition probabilities are described by the transition density

(i)

(ii) $P(X_{s+t} \in A \mid X_s = x) =$ for any $x \in \mathbb{R}$, $A \subset \mathbb{R}$

density of X_{s+t} given $X_s = x$

BM as a continuous time continuous space Markov process

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM.
Then $(B_t)_{t\geq 0}$ is a with transition
density

Informal explanation: Independent stationary increments imply that $(B_t)_{t\geq 0}$ is Markov with stationary transition density. Given $B_s = x$,
information before time s is irrelevant.

$$P(B_{s+t} \leq u \mid B_s = x) =$$

BM as a continuous time continuous space Markov process

Let $t_1 < t_2 < \cdots < t_n < \infty$, $(a_i, b_i) \subset \mathbb{R}$. Then

$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)) =$

$=$

$=$

$=$

More generally,

$$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2), \dots, B_{t_n} \in (a_n, b_n))$$
$$= \int \cdots \int_{(a_1, b_1) \times \cdots \times (a_n, b_n)} p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \cdots dx_n$$

Diffusion equation. Transition semigroup. Generator

Let $(X_t)_{t\geq 0}$ be a Markov process,

Suppose we want to know how the distribution of X_t evolves in time:

CK

We call $(P_t)_{t\geq 0}$ the transition semigroup $\left[P_{s+t} f(x) = P_s(P_t f(x)) \right]$

<u>Proposition</u> Let $(P_t)_{t\geq 0}$ be the transition semigroup of BM.

Then (i) the "infinitesimal generator" of $P(t)$ is given by

(ii) density p_t satisfies [K backward]

(iii) density p_t satisfies [K forward]

↑ diffusion equation