MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next: PK 8.1-8.2

Week 9:

CAPES

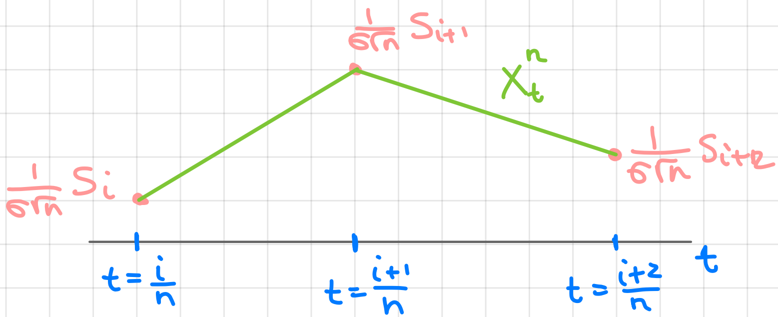
- homework 7 (due Friday, May 27)
- HW6 regrades are active on Gradescope until May 28, 11 PM
- Friday May 27 office hour: AP&M 7321

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote $S_m=\sum_{k=1}^{m}\xi_k$ and define

$$X_t^n=\frac{1}{\sigma\sqrt{n}}\left(S_{[nt]}+(nt-[nt])\,\xi_{[nt]+1}\right)$$



Theorem (Donsker) $(X_t^n)_{t\geq 0}$ converges in distribution to the standard BM.

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$E(\xi_i) = 0$, $Var(\xi_i) = 1$.

Denote $S_m = \sum_{i=1}^{m} \xi_i$, $S_0 = 0$. $(S_m)_{m\geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$ $P(S \text{ reaches } -a \text{ before } b) = \frac{b}{a+b}$.

If X_t^n is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) = P(S \text{ hits } -\sqrt{n}\, a \text{ before } \sqrt{n}\, b)$$

$$= \frac{\sqrt{n}\, b}{\sqrt{n}\, a + \sqrt{n}\, b} = \frac{b}{a+b}$$

$$\Rightarrow P(B \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$$

$$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty},\ E(\tilde{\xi}_i) = 0,\ Var(\tilde{\xi}_i) = 1,\ P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|) < \infty$ $\forall t \geq 0$ and

$$E(X_t \mid \{X_u, 0 \leq u \leq s\}) = X_s$$

Proposition Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) $(B_t)_{t\geq 0}$ is a martingale

(ii) $(B_t^2 - t)_{t\geq 0}$ is a martingale (w.r.t. $(B_t)_{t\geq 0}$)

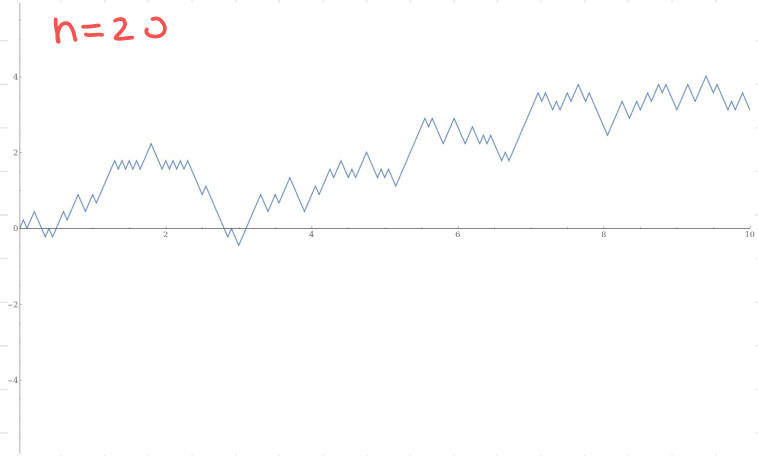
"Proof": $E(B_t \mid \{B_u, 0 \leq u \leq s\}) = E(B_s + B_t - B_s \mid \{B_u, 0 \leq u \leq s\}) = B_s + 0 = B_s$

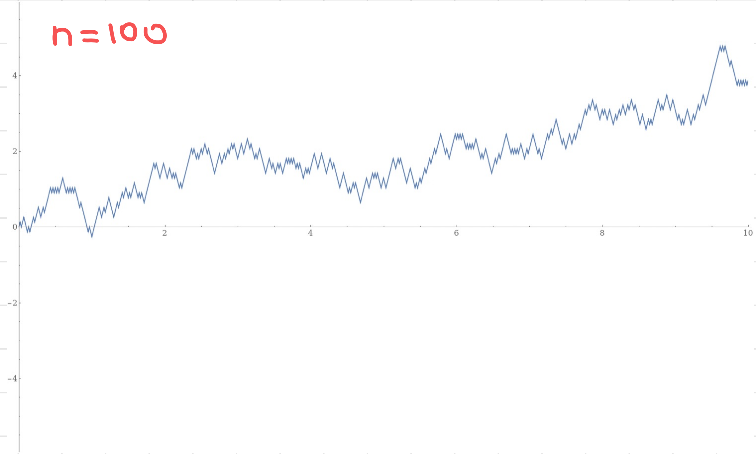
$$E(B_t^2 - t \mid \{B_u, 0 \leq u \leq s\}) = E(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 - t \mid \{B_u, 0 \leq u \leq s\})$$

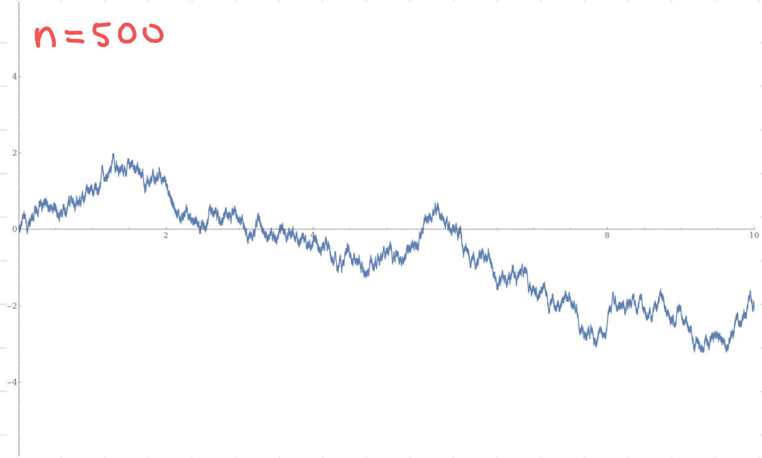
$$= B_s^2 + 0 + t - s - t = B_s^2 - s$$

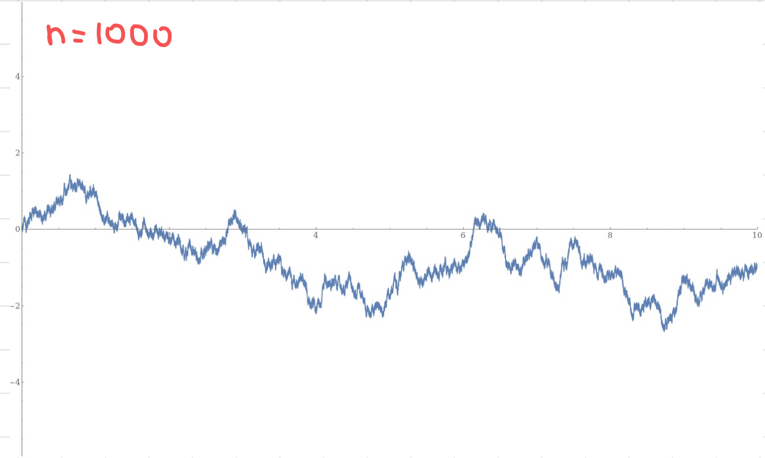
Thm (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2 - t)_{t\geq 0}$ is a martingale. Then $(X_t)_{t\geq 0}$ is a BM.

Approximating a BM with random walks X_t^n









Stopping times and the strong Markov property (lec.?)

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time

2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec. ?)

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = x$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

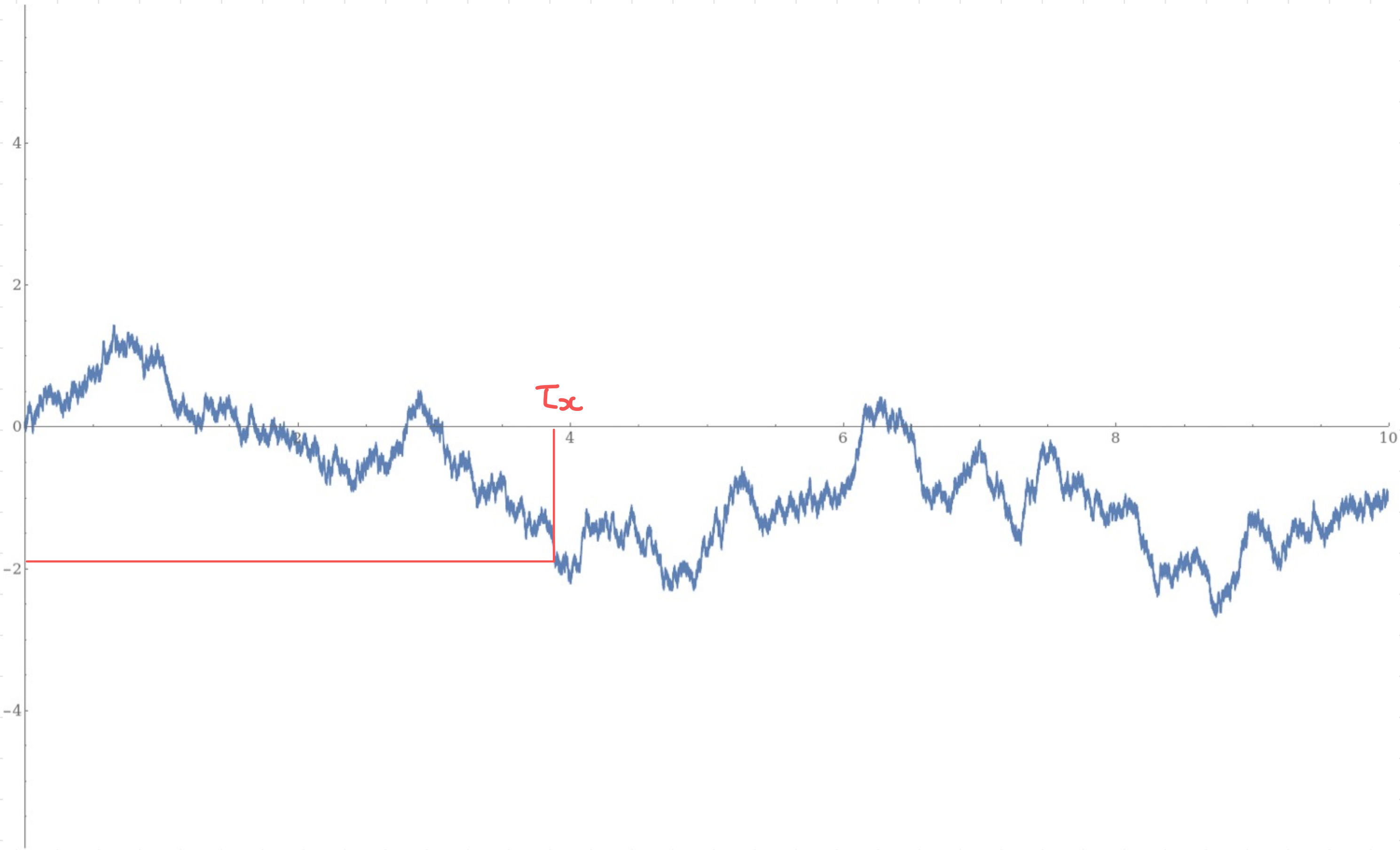
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from x

Example $(B_t)_{t\geq 0}$ is Markov. For any $x\in\mathbb{R}$ define

$\tau_x = \min\{t : B_t = x\}$. Then

- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is a BM starting from 0
- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is independent of $\{B_s,\ 0\leq s\leq \tau_x\}$

(independent of what B was doing before it hit x)



4
2
0
-2
-4
2
4
6
8
10
τ_x

Reflection principle

<u>Thm</u>. Let $(B_t)_{t\ge 0}$ be a standard BM. Then for any $t\ge 0$ and $x>0$

$\cancel{(S_t)_{t\ge 0} \overset{(d)}{=} (|B_t|)_{t\ge 0}}$

$$P\Big(\underbrace{\max_{0\le u\le t} B_u}_{=S_t} > x\Big) = P(|B_t|>x) = 2P(B_t>x)$$

<u>Proof</u>. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0\le u\le \tau_x\}$

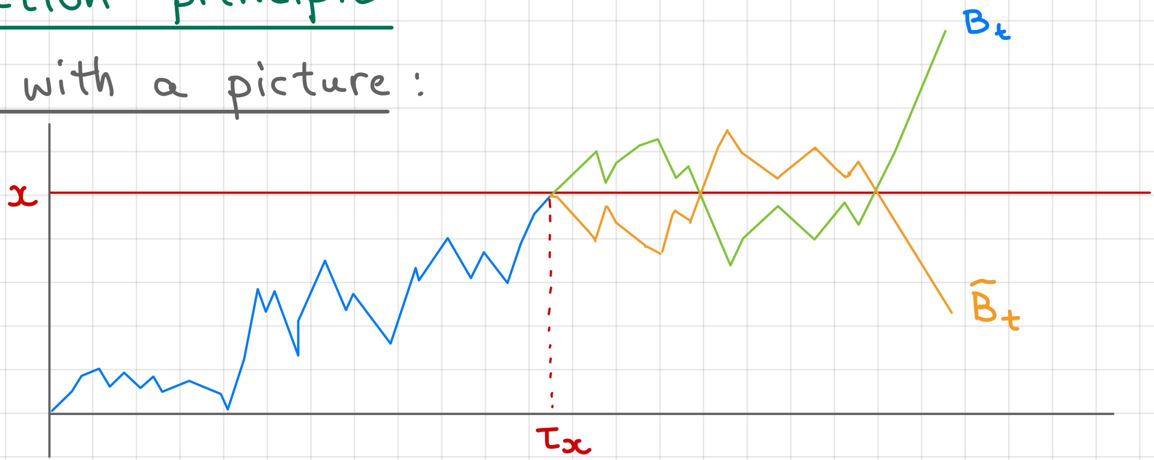
From the definition of τ_x, $\max\limits_{0\le u\le t} B_u \ge x \Leftrightarrow \tau_x \le t$. Then

$$P\Big(\max_{0\le u\le t} B_u \ge x,\ B_t < x\Big) =$$

Now $P\Big(\max\limits_{0\le u\le t} B_u \ge x\Big) =$

Reflection principle

Proof with a picture:



If $(B_t)_{t\geq 0}$ is a BM, then $(\tilde{B}_t)_{t\geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t , & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}) , & t > \tau_x \end{cases}$$

$$P(\tau_x \leq t) = P(\tau_x \leq t, B_t > x) + P(\tau_x \leq t, B_t < x)$$

$\Rightarrow$ to each sample path with $\max_{0\leq u\leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max_{0\leq u \leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$P(\max_{0\leq u\leq t} B_u \geq x, B_t > x)$
$\shortparallel$

$$P(\max_{0\leq u\leq t} B_u \geq x, B_t < x) = P(B_t > x) \Rightarrow P(\max_{0\leq u\leq t} B_u \geq x) = 2P(B_t \geq x) \quad \blacksquare$$