MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next: PK 8.1-8.2

Week 10:

- homework 8 (due Friday, June 3)
- HW7 regrades are active on Gradescope until June 4, 11 PM
- homework 9 and solutions are available on the course website

Reflected BM

<u>Def</u>. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$R_t = |B_t| = \begin{cases} B_t, & \text{if } B_t \geq 0 \\ -B_t, & \text{if } B_t < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

<u>Moments</u>: $E(R_t) = \int_{-\infty}^{+\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \underset{\text{symmetry}}{=} 2\int_0^{\infty} \frac{x e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \sqrt{\frac{2t}{\pi}}$

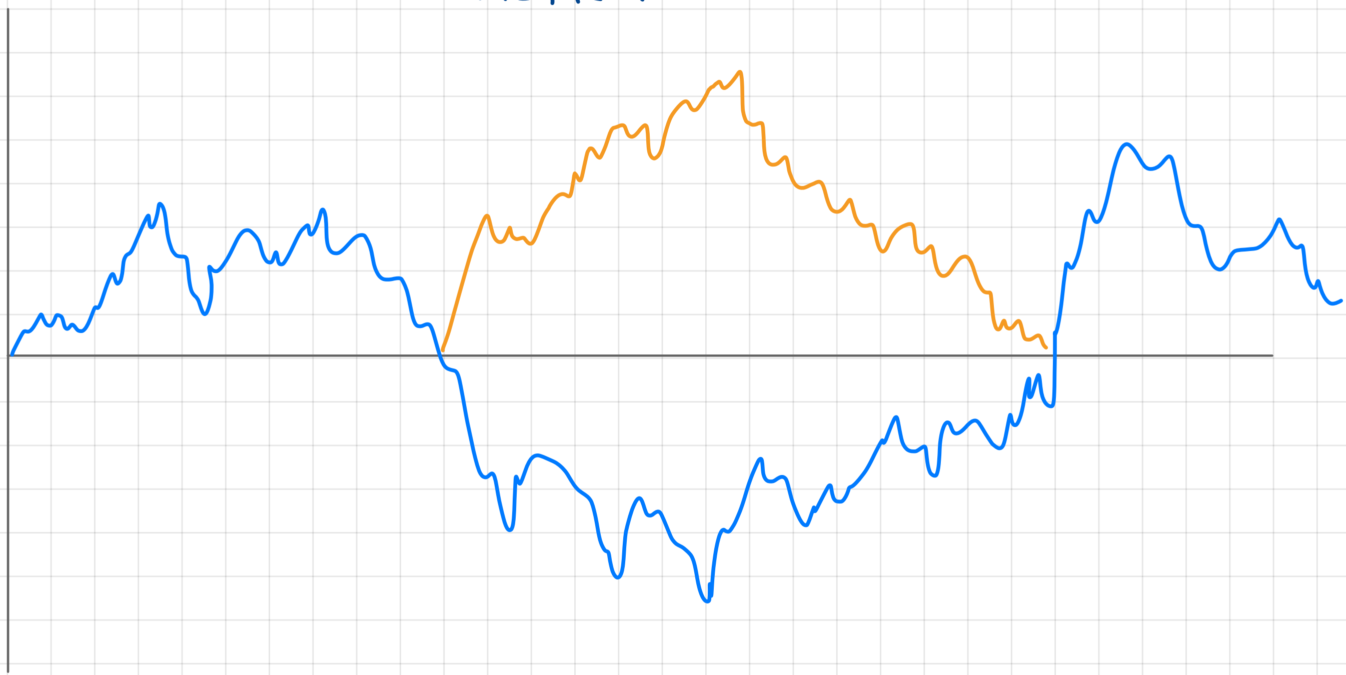
$$Var(R_t) = E(B_t^2) - \left(E(|B_t|)\right)^2 = t - \frac{2t}{\pi} = \left(1 - \frac{2}{\pi}\right)t$$

<u>Transition density</u>: $P(R_t \leq y \mid R_0 = x) = P(-y \leq B_t \leq y \mid B_0 = x)$

$$= \int_{-y}^{y} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-s)^2}{2t}} ds \Rightarrow p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}} \right)$$

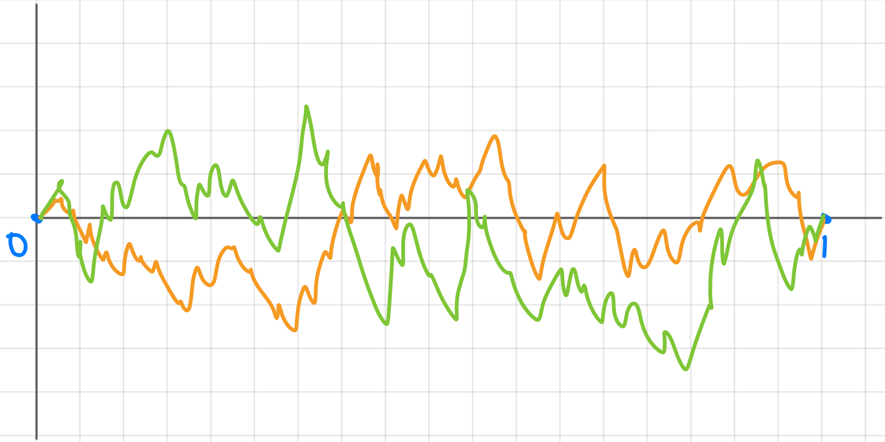
<u>Thm</u> (Lévy) Let $M_t = \max_{0 \leq u \leq t} B_u$. Then $(M_t - B_t)_{t\geq 0}$ is a reflected BM.

Reflected BM



Brownian bridge

Brownian bridge is constructed from a BM by conditioning on the event $\{B(0)=0, B(1)=0\}$.



Thm 1. Brownian bridge is a continuous Gaussian process on $[0,1]$ with mean 0 and covariance function

$\Gamma(s,t) =$

Brownian motion with drift

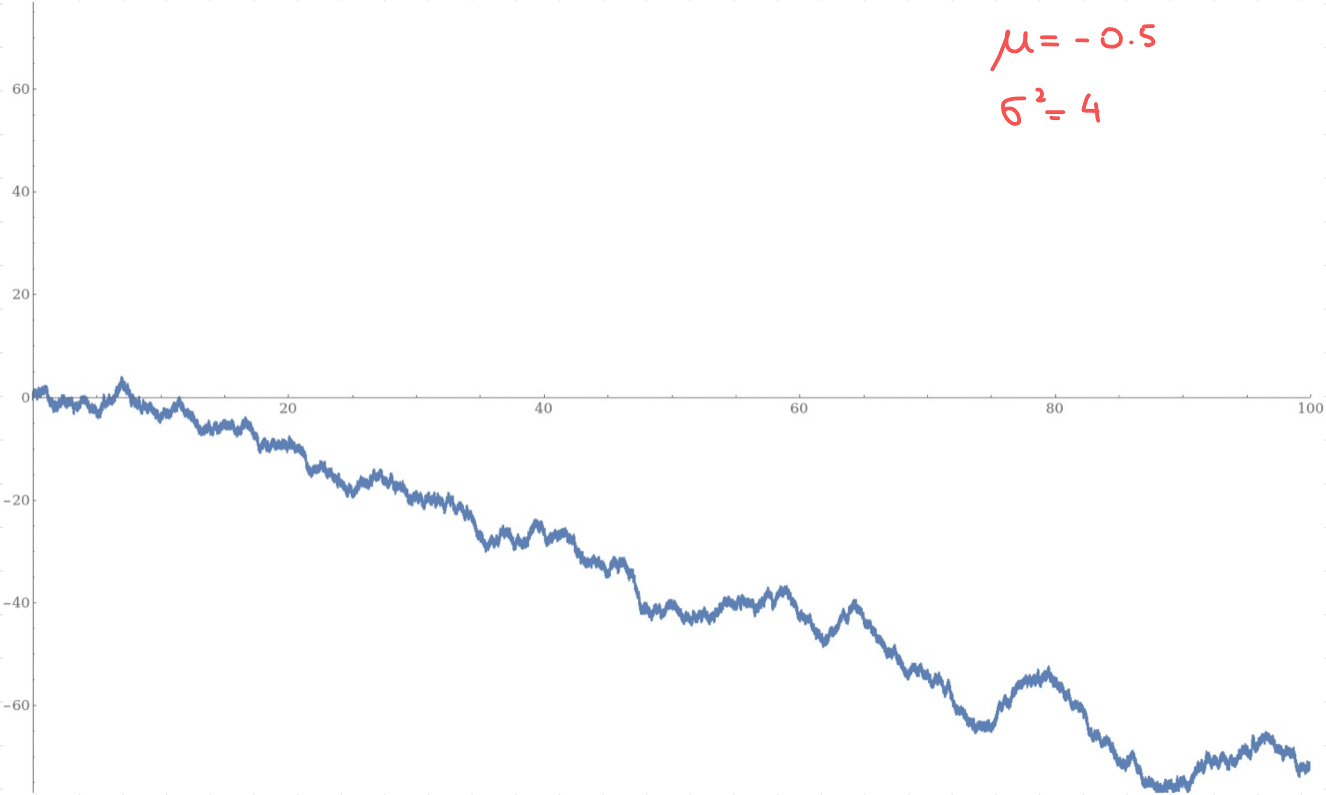
<u>Def</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then for $\mu\in\mathbb{R}$ and $\sigma>0$ the process $(X_t)_{t\geq 0}$ with $X_t =$, $t\geq 0$ is called the Brownian motion with drift μ and variance paremeter σ^2.

<u>Remark</u> BM with drift μ and variance paremeter σ is a stochastic process $(X_t)_{t\geq 0}$ satisfying

1) $X_0 = 0$, $(X_t)_{t\geq 0}$ has continuous sample paths
2) $(X_t)_{t\geq 0}$ has independent increments
3) For $t>s$ $X_t - X_s \sim$

In particular, $X_t \sim$ $\Rightarrow X_t$ is not centered, not symmetric w.r.t. the origin

Brownian motion with drift



Gambler's ruin problem for BM with drift

Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$. Fix $a < x < b$ and denote

$$T = T_{ab} = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}, \text{ and}$$

$$u(x) = P(X_T = b \mid X_0 = x).$$

Theorem.

(i) $u(x) =$

(ii) $E(T_{ab} \mid X_0 = x) =$

No proof

Example

Fluctuations of the price of a certain share is modeled by the BM with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

(a) What is the probability that you will sell at profit?

(b) What is the expected time until you sell the share?

Denote by $(X_t)_{t \geq 0}$ a BM with drift $\frac{1}{10}$ and variance 4,

$x =$, $b =$, $a =$. Then $2\mu/\sigma^2 =$ and

(a) $P(X_T = 110 \mid X_0 = 100) =$

(b) $E(T \mid X_0 = 100) =$

Maximum of a BM with negative drift

Thm Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu<0$, variance σ^2 and $X_0=0$. Denote $M=\max_{t\geq 0} X_t$. Then

Proof. $X_0=0$, therefore $M\geq 0$. For any $b>0$

$$P(M>b) =$$

$$=$$

$$=$$

$$P(M>b)=$$

Geometric BM

Def. Stochastic process $(Z_t)_{t\geq 0}$ is called a geometric Brownian motion with drift parameter α and variance σ^2 if $X_t =$ is a BM with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance σ^2.

In other words, $Z_t =$, where $(B_t)_{t\geq 0}$ is a standard BM and $z>0$ is the starting point $Z_0 = z$.

If $0 \leq t_1 < t_2 < \dots < t_n$, then $\frac{Z_{t_i}}{Z_{t_{i-1}}} =$

Since B has independent increments

$\frac{Z_{t_1}}{Z_{t_0}}, \frac{Z_{t_2}}{Z_{t_1}}, \dots, \frac{Z_{t_n}}{Z_{t_{n-1}}}$ are independent and

$\frac{Z_{t_n}}{Z_{t_0}} =$ $\leftarrow$ "relative change of price = product of independent relative changes"

Expectation of Geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ.

Then

$$E(Z_t \mid Z_0 = z) =$$

$$E(e^{\sigma B_t}) =$$

$$\Rightarrow E(Z_t \mid Z_0 = z) = z\, e^{(\alpha - \frac{1}{2}\sigma^2)t}\, e^{t\frac{\sigma^2}{2}} =$$

Remark

It can be shown that for $0 < \alpha < \frac{1}{2}\sigma^2$ $Z_t \to 0$ as $t \to \infty$

At the same time, for $\alpha > 0$ $E(Z_t) \to \infty$.

Variance of geometric BM

$E(Z_t^2 \mid Z_0 = z) =$

$=$

$Var(Z_t \mid Z_0 = z) =$

Theorem.

Let $(Z_t)_{t \geq 0}$ be geometric BM with paremeters α and σ^2.

Then

(i) $E(Z_t \mid Z_0 = z) = z e^{\alpha t}$

(ii) $Var(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t} \left(e^{\sigma^2 t} - 1 \right)$

Gambler's ruin for geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Let $A < 1 < B$, and denote $T = \min\{t : \frac{Z_t}{Z_0} = A$ or $\frac{Z_t}{Z_0} = B\}$.

Theorem

$$P\left(\frac{Z_T}{Z_0} = B\right) =$$

Example Fluctuations of the price are modeled by a geometric BM with drift $\alpha = 0.1$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

Take $A =$, $B =$, $2\alpha/\sigma^2 =$, $1 - 2\alpha/\sigma^2 =$

$$P(X_T = 110 \mid X_0 = 100) =$$