MATH 285: Stochastic Processes

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Today: Introduction.
Definition of Markov processes

- Test Homework on Gradescope

Stochastic Processes

Def. 1.1 Let T and S be two sets, and let $(\Omega, \mathbb{P})$ be a probability space. We call a collection $(X_t)_{t \in T}$ of random variables that are all defined on the same probability space $(\Omega, \mathbb{P})$ and take values in S a stochastic process indexed by T and taking values in S. If $T = [0, +\infty)$, then $(X_t)_{t \geq 0}$ is called a continuous time $[0,1]$ stochastic process. If $T = \mathbb{N}$, then $(X_n)_{n \in \mathbb{N}}$ is a discrete time stochastic process.

T : index set (time), S: state space

$$X : \Omega \times T \to S \qquad X_t(\omega) \in S$$

Stochastic Processes

Motivation: Mathematical model of phenomena that evolve in time in a random way

Stochastic processes have applications in many disciplines such as biology,[6] chemistry,[7] ecology,[8] neuroscience,[9] physics,[10] image processing, signal processing,[11] control theory,[12] information theory,[13] computer science,[14] cryptography[15] and telecommunications.[16]

en.m.wikipedia.org

+ finance

Prices, sizes of populations, number of particles, ...

Examples

<u>Example 1.2</u> $X_1, X_2, \dots$ are i.i.d. random variables (real-valued) defined on the same probability space. Then $(X_n)_{n\in\mathbb{N}}$ is a discrete time stochastic proeess.

$$T = \mathbb{N}, \quad S = \mathbb{R}$$

Define $S_n := X_1 + X_2 + \dots + X_n$. Then $(S_n)_{n\in\mathbb{N}}$ is again a discrete time stochastic process

<u>Example 1.3</u> As above, but $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$

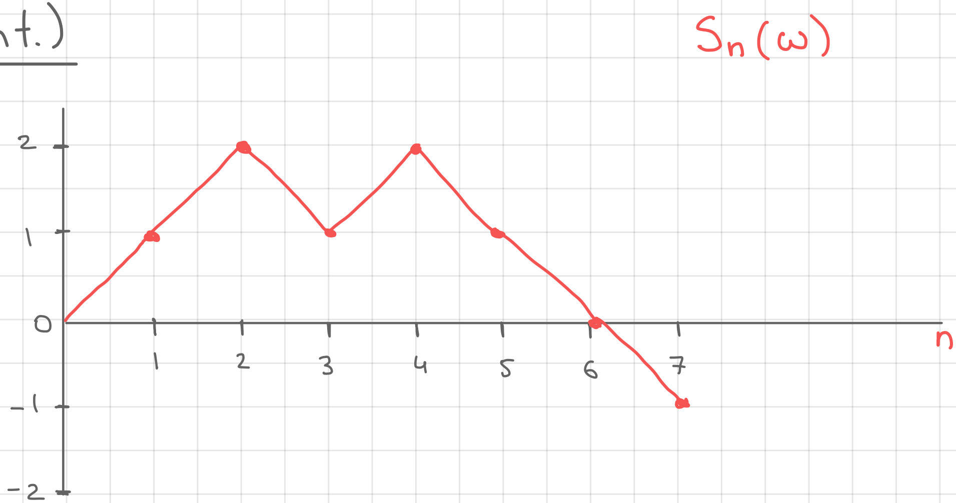
(X_n): $T = \mathbb{N}, \quad S = \{-1, 1\}$

(S_n): $T = \mathbb{N}, \quad S = \mathbb{Z}$ (symmetric random walk on $\mathbb{Z}$)

Examples

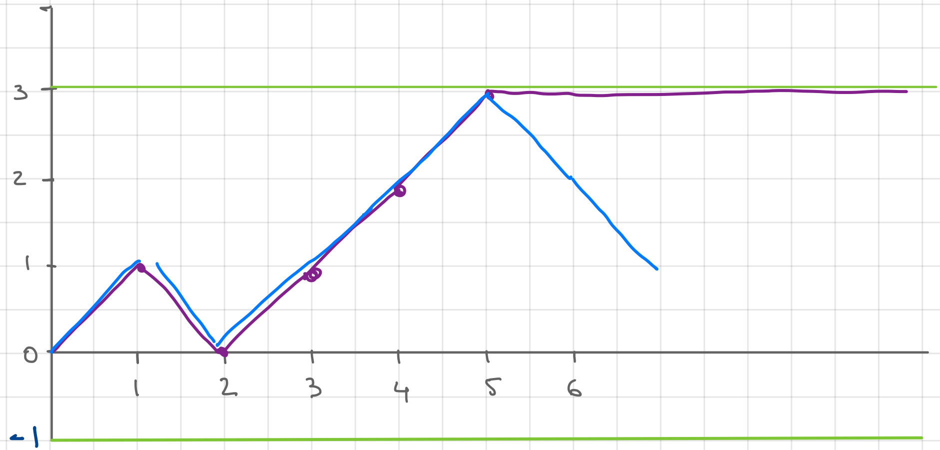
Example 1.3 (cont.)

Random walk



Example 1.4

- reflected RW
- absorbing RW
- partially reflected RW



Discrete time Markov chain

Suppose that S is a discrete state space, and $(X_1, \dots, X_n)$ is a collection of r.v.s with values in S.

Q: $\mathbb{P}[(X_1, \dots, X_n) = (i_1, \dots, i_n)]$ for $(i_1, \dots, i_n) \in S^n$

$$\mathbb{P}[X_1 = i_1, X_2 = i_2, \dots, X_n = i_n]$$

$$= \mathbb{P}[X_n = i_n \mid X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}] \cdot \mathbb{P}[X_1 = i_1, \dots, X_{n-1} = i_{n-1}]$$

$$= \mathbb{P}[X_n = i_n \mid X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}] \times$$

$$\times \mathbb{P}[X_{n-1} = i_{n-1} \mid X_1 = i_1, \dots, X_{n-2} = i_{n-2}] \cdots \mathbb{P}[X_2 = i_2 \mid X_1 = i_1] \mathbb{P}[X_1 = i_1]$$

$$(\text{Markov}) = \mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}] \cdot \mathbb{P}[X_{n-1} = i_{n-1} \mid X_{n-2} = i_{n-2}] \cdots \mathbb{P}[X_2 = i_2 \mid X_1 = i_1] \times$$

$$\times \mathbb{P}[X_1 = i_1]$$

Discrete time Markov chain

Def 1.5 Let X_n be a discrete time stochastic process with state space S that is finite or countably infinite. Then X_n is called a discrete time Markov chain if for each $n \in \mathbb{N}$ and each $(i_1, \dots, i_n) \in S^n$

$$(M) \qquad \mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}]$$

Example 1.2 (Recall $\{X_i\}$ are i.i.d.)

Suppose that S is finite or countably infinite

Then (by independence) $\mathbb{P}[X_n = i_n \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n]$

and $\mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}] = \mathbb{P}[X_n = i_n]$, so (M) is satisfied.

$$\mathbb{P}[X_1 = i_1, \dots, X_n = i_n] = \mathbb{P}[X_1 = i_1, \dots, X_n = i_n]$$