MATH 285: Stochastic Processes

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Today: Kolmogorov’s equations
Poisson processes

- Homework 5 is due on Sunday, February 20, 11:59 PM

Infinitesimal description

Transition rates completely determine the Markov chain.

Q: What is the distribution of X_t? $\mathbb{P}_i[X_t=j]=p_t(i,j)=?$

Thm 19.3 Let $(X_t)_{t\geq 0}$ be a MC with state space S and transition rates $q(i,j)$. Then the transition probabilities satisfy

$$p_t(i,i)=1-q(i)t+o(t) \quad \text{as } t\to 0 \quad \text{for } i\in S$$

$$p_t(i,j)=q(i,j)t+o(t) \quad \text{as } t\to 0 \quad \text{for } i\neq j$$

Proof.

(1) $p_t(i,i)=\mathbb{P}_i[X_t=i]\geq \mathbb{P}_i[J_1>t]=e^{-q(i)t}=1-q(i)t+o(t)$

(2) $p_t(i,j)\geq q(i,j)t+o(t)$

$$p_t(i,j)=\mathbb{P}_i[X_t=j]\geq \mathbb{P}_i[J_1\leq t,\ Y_1=j,\ S_2>t]=\mathbb{P}_i[Y_1=j]\,\mathbb{P}_i[S_1\leq t,\ S_2>t\mid Y_1=j]$$

$$=p(i,j)\left(1-e^{-q(i)t}\right)e^{-q(j)t}=p(i,j)\left[1-(1-q(i)t+o(t))\right](1-q(j)t+o(t))$$

$$=p(i,j)\,q(i)t+o(t)=q(i,j)t+o(t)$$

Infinitesimal description

(3) We can write (1) and (2) as

$$p_t(i,i) \geq 1 - q(i)t + \xi_{ii}(t), \quad \xi_{ii}(t) = o(t)$$

$$p_t(i,j) \geq q(i,j)t + \xi_{ij}(t), \quad \xi_{ij}(t) = o(t)$$

Then

$$p_t(i,i) = 1 - q(i)t + \xi_{ii}(t) + \eta_{ii}(t), \quad \eta_{ii}(t) \geq 0$$

$$p_t(i,j) = q(i,j)t + \xi_{ij}(t) + \eta_{ij}(t), \quad \eta_{ij}(t) \geq 0$$

Take the sum

$$1 = p_t(i,i) + \sum_{j \neq i} p_t(i,j) = 1 + \overbrace{\Big(-q(i) + \sum_{j\neq i} q(i,j)\Big)}^{=0} t + \sum_j \xi_{ij} + \sum_j \eta_{ij}$$

$$\Rightarrow \sum_j \xi_{ij}(t) + \sum_j \eta_{ij}(t) = 0 \quad \Rightarrow \sum_j \eta_{ij}(t) = o(t) \Rightarrow \forall j \ \eta_{ij}(t) = o(t) \quad \blacksquare$$

<u>Remark</u> In order to identify a Markov chain it is enough to compute $p_t(i,j)$ to first order in t as $t \to 0$.

Kolmogorov's Equations

Recall: $p_t(i,j) = \mathbb{P}[X_t = j \mid X_0 = i]$, distribution of X_t

$p_t(i,i) = 1 - q(i)t + o(t)$ as $t \to 0$ for $i \in S$

$p_t(i,j) = q(i,j)t + o(t)$ as $t \to 0$ for $i \neq j$

Def 20.1 Let $(X_t)_{t\geq 0}$ be a continuous-time MC with state space S and transition rates $[q(i,j)]_{i,j\in S}$. Define the infinitesimal generator A given by

$$A_{ij} = \underset{i\neq j}{q(i,j)}, \quad A_{ii} = -q(i) = -\sum_{j\in S} q(i,j)$$

Thm 20.2 Let $(X_t)_{t\geq 0}$ be a continuous-time MC with infinitesimal generator A. Let P_t denote the matrix $[P_t]_{ij} = p_t(i,j)$. Then

$$\frac{d}{dt} P_t = \underset{\uparrow \text{forward}}{P_t A} = \underset{\uparrow \text{backward}}{A P_t} \quad \text{and} \quad P_0 = I$$

Kolmogorov's equations

Proof: Fix $t \geq 0$ and $h > 0$. By the Markov property

- $p_{t+h}(i,j) = \mathbb{P}[X_{t+h} = j \mid X_0 = i] = \sum_{k \in S} \mathbb{P}[X_{t+h} = j \mid X_t = k]\, \mathbb{P}_i[X_t = k]$

  $= \sum_{k \in S} p_t(i,k)\, p_h(k,j)$

- From the infinitesimal description

  $p_{t+h}(i,j) = p_t(i,j)\left(1 - q(j)h + o(h)\right) + \sum_{k \neq j} p_t(i,k)\left(q(k,j)h + o(h)\right)$

- $p_{t+h}(i,j) - p_t(i,j) = \left[p_t(i,j)\left(-q(j)\right) + \sum_{k \neq j} p_t(i,k)\, q(k,j)\right]h + o(h)$

  $= \left[\sum_k p_t(i,k) A_{kj}\right]h + o(h) = [P_t A]_{ij}\, h + o(h)$

- $\lim_{h \to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h} = \frac{d}{dt}[P_t]_{ij} = [P_t A]_{ij}$

- Backward equation: $\mathbb{P}[X_{t+h} = j \mid X_0 = i] = \sum_{k \in S} \mathbb{P}[X_{t+h} = j \mid X_h = k]\, \mathbb{P}_i[X_h = k]$

Q-matrices and Matrix exponentials

Let S be a finite set and let $Q = (q_{ij})_{i,j \in S}$

Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^Q$, series converges

- Generally speaking, $e^{Q_1 + Q_2} \neq e^{Q_1} e^{Q_2}$ (true if Q_1 and Q_2 commute)

Thm Let Q be a matrix. Set $P_t = e^{tQ}$, $t \geq 0$. Then

(i) $P_{s+t} = P_s P_t$ for all s, t (semigroup property)

(ii) $(P_t)_{t \geq 0}$ is the unique solution to the forward equation

$$\frac{d}{dt} P_t = P_t Q \quad , \quad P_0 = I$$

(iii) $(P_t)_{t \geq 0}$ is the unique solution to the backward equation

$$\frac{d}{dt} P_t = Q P_t \quad , \quad P_0 = I$$

(iv) for $k = 0, 1, 2, \ldots$ $\left(\frac{d}{dt}\right)^k \Big|_{t=0} P_t = Q^k$

Q-matrices and Matrix exponentials

We say that Q is a Q-matrix if

- $0 \leq -Q_{ii} < \infty$ for all $i \in S$
- $Q_{ij} \geq 0$ for all $i \neq j$
- $\sum_{j \in S} Q_{ij} = 0$

Thm Matrix Q is a Q-matrix if and only if

$$P_t = e^{tQ} \text{ is a stochastic matrix for all } t \geq 0.$$

Three equivalent descriptions of a continuous-time MC

Let (X_t) be a right-continuous process on S (finite), let A be a Q-matrix
↑ generator

The following conditions are equivalent:

(jump and hold) jump chain (Y_n) is a MC with $p(Y_1 = j \mid Y_0 = i) = \frac{A_{ij}}{-A_{ii}}$ and given $Y_{n-1} = i$, the sojourn times (S_n) satisfy $S_n \sim \mathrm{Exp}(-A_{ii})$

(infinitesimal) (X_t) is Markov and $p_h(i,j) = \delta_{ij} + A_{ij} h + o(h)$

(transition probabilities) (X_t) is Markov and $P_t = e^{tA}$ for $t \geq 0$

Examples

Consider $(X_t)_{t\geq 0}$ with $S=\{0,1\}$

$$\begin{tikzcd} 0 \arrow[r, bend left, "\alpha"] & 1 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

$\alpha>0,\ \beta>0$

and generator

$$A=\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$ Compute P_t.

- $$A^2=\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}=\begin{bmatrix} \alpha^2+\alpha\beta & -\alpha^2-\alpha\beta \\ -\alpha\beta-\beta^2 & \alpha\beta-\beta^2 \end{bmatrix}=-(\alpha+\beta)\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$
- $A^k=(-1)^{k-1}(\alpha+\beta)^{k-1}A$, and thus
- $$e^{tA}=\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}=I+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\alpha+\beta)^{k-1}t^k}{k!}A=I-\frac{1}{\alpha+\beta}\left(e^{-(\alpha+\beta)t}-1\right)A$$
- $$\mathbb{P}[X_t=1\mid X_0=0]=\frac{1}{\alpha+\beta}\left(1-e^{-(\alpha+\beta)t}\right)\cdot\alpha$$

Example : Poisson process

Let (X_t) be a Poisson process:

$$q(i,j) = \lambda \delta_{ij}$$
$$q(i) = \lambda$$

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 4 \xrightarrow{\lambda} 5 \xrightarrow{\lambda} 6 \; \cdots$$

$$P_t \begin{bmatrix} \cdot & & \\ 0 & \cdot & \\ & & \cdot \end{bmatrix} \qquad A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & \cdots \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}$$

Compute $p_t(i,j)$

- (X_t) is increasing, so $p_t(i,j) = 0$, $j < i$
- Write the forward equation "$\frac{d}{dt} P_t = P_t A$" for $p_t(i,i)$:

$$p_t'(i,i) = \sum_k p_t(i,k) A_{ki} = -\lambda\, p_t(i,i), \qquad p_0(i,i) = 1$$

so $p_t(i,i) = e^{-\lambda t}$ for all i

Example : Poisson process

- for $i<j$ $\quad p_t'(i,j) = \sum_k p_t(i,k)A_{kj} = \lambda p_t(i,j-1) - \lambda p_t(i,j)$, $p_0(i,j)=0$

so $\quad \lambda p_t(i,j) + p_t'(i,j) = \lambda p_t(i,j-1)$. Now consider

$$\left(e^{\lambda t} p_t(i,j)\right)' = \lambda e^{\lambda t} p_t(i,j) + e^{\lambda t} p_t'(i,j) = e^{\lambda t}\left(\lambda p_t(i,j) + p_t'(i,j)\right)$$

$$= e^{\lambda t} \lambda\, p_t(i,j-1)$$

- Induction: $\left(e^{\lambda t} p_t(i,i+1)\right)' = e^{\lambda t} \lambda\, p_t(i,i) = \lambda \Rightarrow e^{\lambda t} p_t(i,i+1) = \lambda t + C$

$p_0(i,i+1) = 0 \Rightarrow C = 0$, thus $p_t(i,i+1) = \lambda t\, e^{-\lambda t}$

If $\quad P_t(i,i+k) = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}$, then

$$\left(e^{\lambda t} P_t(i,i+k+1)\right)' = e^{\lambda t} \lambda P_t(i,i+k) = \frac{\lambda^{k+1} t^k}{k!} \Rightarrow e^{\lambda t} P_t(i,i+k+1) = \frac{(\lambda t)^{k+1}}{(k+1)!}.$$

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