# MATH 285: Stochastic Processes

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## Today: Kolmogorov’s equations
## Poisson processes

- Homework 5 is due on Sunday, February 20, 11:59 PM

## Infinitesimal description

Transition rates completely determine the Markov chain.

Q: What is the distribution of $X_t$? $\mathbb{P}_i[X_t=j] = p_t(i,j) = ?$

**Thm 19.3** Let $(X_t)_{t\geq 0}$ be a MC with state space $S$ and transition rates $q(i,j)$. Then the transition probabilities satisfy

$$p_t(i,i) = 1 - q(i)t + o(t) \quad \text{as } t\to 0 \quad \text{for } i\in S$$

$$p_t(i,j) = q(i,j)t + o(t) \quad \text{as } t\to 0 \quad \text{for } i\neq j$$

**Proof.**

(1) $p_t(i,i) = \mathbb{P}_i[X_t = i]$

(2) $p_t(i,j)$

$p_t(i,j) = \mathbb{P}_i[X_t = j] \geq$

$=$

$=$

# Infinitesimal description

(3) We can write (1) and (2) as

$$p_t(i,i) \geq 1 - q(i)t + \xi_{ii}(t), \quad \xi_{ii}(t) = o(t)$$

$$p_t(i,j) \geq q(i,j)t + \xi_{ij}(t), \quad \xi_{ij}(t) = o(t)$$

Then

$$p_t(i,i) = 1 - q(i)t + \xi_{ii}(t)$$

$$p_t(i,j) = q(i,j)t + \xi_{ij}(t)$$

Take the sum

$$p_t(i,i) + \sum_{j \neq i} p_t(i,j) =$$

$\Rightarrow$ $\qquad\qquad$ $\Rightarrow$ $\qquad\qquad$ $\Rightarrow$

<u>Remark</u> In order to identify a Markov chain it is enough to compute $p_t(i,j)$ to first order in $t$ as $t \to 0$.

# Kolmogorov's Equations

Recall: $p_t(i,j) = \mathbb{P}[X_t = j \mid X_0 = i]$, distribution of $X_t$

$p_t(i,i) = 1 - q(i)t + o(t)$ as $t \to 0$ for $i \in S$

$p_t(i,j) = q(i,j)t + o(t)$ as $t \to 0$ for $i \neq j$

**Def 20.1** Let $(X_t)_{t \geq 0}$ be a continuous-time MC with state space $S$ and transition rates $[q(i,j)]_{i,j \in S}$. Define the infinitesimal generator $A$ given by

$$A_{ij} = \qquad\qquad , \quad A_{ii} =$$

**Thm 20.2** Let $(X_t)_{t \geq 0}$ be a continuous-time MC with infinitesimal generator $A$. Let $P_t$ denote the matrix $[P_t]_{ij} = p_t(i,j)$.

Then $\frac{d}{dt} P_t =$ $\qquad\qquad$ and $\quad P_0 =$

# Kolmogorov's equations

**Proof**: Fix $t \geq 0$ and $h > 0$. By the Markov property

- $p_{t+h}(i,j) = \mathbb{P}[X_{t+h} = j \mid X_0 = i] =$

  $=$

- From the infinitesimal description

  $p_{t+h}(i,j) =$

- $p_{t+h}(i,j) - p_t(i,j) =$

  $=$

- $\lim_{h \to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h} = \frac{d}{dt}[P_t]_{ij} =$

- Backward equation: $\mathbb{P}[X_{t+h} = j \mid X_0 = i] =$

# Q-matrices and Matrix exponentials

Let $S$ be a finite set and let $Q = (q_{ij})_{i,j \in S}$

Then the series

- Generally speaking, $e^{Q_1 + Q_2} \neq e^{Q_1} e^{Q_2}$ (true if $Q_1$ and $Q_2$ commute)

<u>Thm</u> Let $Q$ be a matrix. Set , $t \geq 0$. Then

(i) $P_{s+t} =$ for all $s, t$ (semigroup property)

(ii) $(P_t)_{t \geq 0}$ is the unique solution to the forward equation

$$\frac{d}{dt} P_t =$$

(iii) $(P_t)_{t \geq 0}$ is the unique solution to the backward equation

$$\frac{d}{dt} P_t =$$

(iv) for $k = 0, 1, 2, \ldots$ $\left(\frac{d}{dt}\right)^k \Big|_{t=0} P_t =$

## Q-matrices and Matrix exponentials

We say that $Q$ is a $Q$-matrix if

- $-Q_{ii}$ for all $i \in S$
- $Q_{ij}$ for all $i \neq j$
- $\sum_{j \in S} Q_{ij}$

<u>Thm</u> Matrix $Q$ is a $Q$-matrix if and only if

is a stochastic matrix for all $t \geq 0$.

### Three equivalent descriptions of a continuous-time MC

Let $(X_t)$ be a right-continuous process on $S$ (finite), let $A$ be a $Q$-matrix

↑ generator

The following conditions are equivalent:

(jump and hold) jump chain $(Y_n)$ is a MC with $p(Y_1 = j \mid Y_0 = i) = \frac{A_{ij}}{-A_{ii}}$ and given $Y_{n-1} = i$, the sojourn times $(S_n)$ satisfy $S_n \sim \mathrm{Exp}(-A_{ii})$

(infinitesimal) $(X_t)$ is Markov and $p_h(i,j) = \delta_{ij} + A_{ij}\, h + o(h)$

(transition probabilities) $(X_t)$ is Markov and $P_t = e^{tA}$ for $t \geq 0$

# Examples

Consider $(X_t)_{t\geq 0}$ with $S=\{0,1\}$ and generator

$$
\begin{tikzcd}
0 \arrow[r, bend left, "\alpha"] & 1 \arrow[l, bend left, "\beta"]
\end{tikzcd}
$$

$\alpha>0,\ \beta>0$

$$A=\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$ Compute $P_t$.

- $A^2=\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}=\begin{bmatrix} \alpha^2+\alpha\beta & -\alpha^2-\alpha\beta \\ -\alpha\beta-\beta^2 & \alpha\beta-\beta^2 \end{bmatrix}=-(\alpha+\beta)\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$
- $A^k=$ , and thus
- $e^{tA}=\sum_{k=0}^{\infty}\frac{t^k A^k}{k!}=$ $=$
- $\mathbb{P}[X_t=1\mid X_0=0]=$

# Example : Poisson process

Let $(X_t)$ be a Poisson process:

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 4 \xrightarrow{\lambda} 5 \xrightarrow{\lambda} 6 \; \cdots$$

$$q(i,j) = \lambda \delta_{ij}$$

$$q(i) = \lambda$$

$$A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & \cdots \\ & & & \ddots & \ddots \end{bmatrix}$$

Compute $p_t(i,j)$

- $(X_t)$ is increasing, so

- Write the forward equation "$\frac{d}{dt} P_t = P_t A$" for $p_t(i,i)$ :

$$p_t'(i,i) = \sum_k p_t(i,k) A_{ki} = \qquad , \qquad p_0(i,i) =$$

so $p_t(i,i) =$ for all $i$

# Example : Poisson process

- for $i<j$ $\quad p_t'(i,j) = \sum_k p_t(i,k) A_{kj} =$ $\qquad$ , $p_0(i,j) =$

  so $\qquad$ . Now consider

- Induction: $\left(e^{\lambda t} p_t(i,i+1)\right)' = e^{\lambda t} \lambda p_t(i,i) = \lambda \Rightarrow e^{\lambda t} p_t(i,i+1) =$

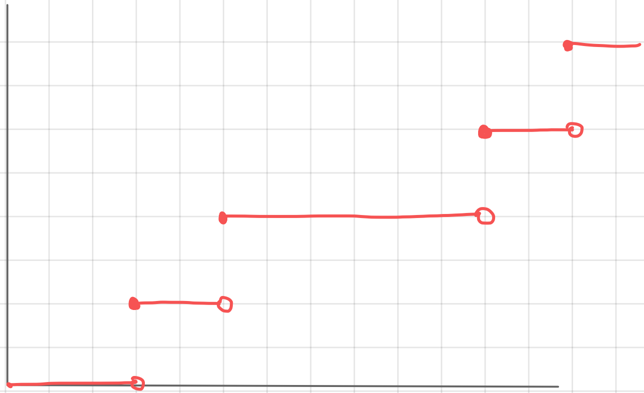
  $p_0(i,i+1) = 0 \Rightarrow C = 0$ , thus $p_t(i,i+1) =$

  If $\quad p_0(i,i+k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$ , then

  $\left(e^{\lambda t} p_t(i,i+k+1)\right)' = e^{\lambda t} \lambda p_t(i,i+k) =$ $\qquad \Rightarrow e^{\lambda t} p_t(i,i+k+1) =$

# Poisson processes

! The jump chain of a Poisson process has a deterministic trajectory



By Prop. 19.2, given the trajectory the sojourn time are independent exponential r.v. with $S_k \sim Exp(q(Y_{k-1}))$

$$\mathbb{P}[S_1 > s_1, \dots, S_n > s_n] = \sum_{i_0, \dots, i_n} \mathbb{P}[S_1 > s_1, \dots, S_n > s_n \mid Y_0 = i_0, \dots, Y_n = i_n] \, \mathbb{P}[Y_0 = i_0, \dots, Y_n = i_n]$$

$=$

$=$

<u>Prop 20.6</u> If $(X_t)$ is a Poisson process, then $S_1, S_2, \dots$ are

# Poisson processes

Alternative construction of a Poisson process (with $X_0 = 0$):

- take a collection of i.i.d. random variables $S_k$, $S_k \sim \mathrm{Exp}(\lambda)$
- define the jump times $J_n = S_1 + \cdots + S_n$, $J_0 = 0$
- set $X_t = n$ for $J_n \le t < J_{n+1}$

Then $X_t$ is a Poisson process with rate $\lambda$.

You can think about $J_n$ as the times of some events, and $X_t$ as the number of events that happend up to time $t$.

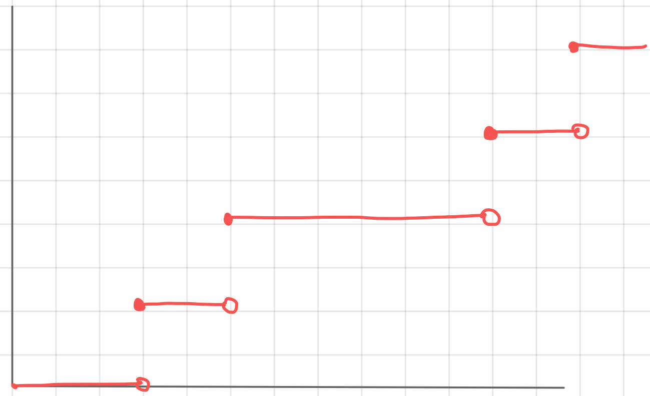
Theorem 20.7 Let $(X_t)_{t \ge 0}$ be a Poisson process of rate $\lambda$, $X_0 = 0$. Then for any $s \ge 0$ the process

is a Poisson process of rate $\lambda$, independent of $\{X_u : 0 \le u \le s\}$

No proof.

# Independent increments

Given a stochastic process $(X_t)_{t\geq 0}$ its increments are random variables



Suppose that $(X_t)$ is a counting process, i.e.,

(jump times = event times, $X_t$ = # of events that occurred up to time $t$). Then for $s<t$

$X_t - X_s$ = # of events that occurred on $(s,t]$.

**Cor. 20.8** If $(X_t)$ is a Poisson process with rate $\lambda$, then for any $0 \leq t_0 < t_1 < \cdots < t_n$ the increments $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$ are independent, and each increment $X_t - X_s$ is a Poisson random variable with rate . These properties uniquely characterize the Poisson process.

# Independent increments

Proof.

- $X_t - X_s = X_{s+(t-s)} - X_s \sim$
- $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent

Induction: Suppose $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent

By Thm 20.7, for any $t \geq 0$ the process

is independent of $X_s$ for $s \leq t_n$

Therefore, $\tilde{X}_t$ is independent of $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$,

and for any $t_{n+1} > t_n$ $\tilde{X}_{t_{n+1}-t_n} = X_{t_{n+1}} - X_{t_n}$ is independent

of $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$

- Independent increments uniquely determine the joint distribution of $(X_{t_0}, \dots, X_{t_n})$ for any $0 \leq t_0 < \dots < t_n < \infty$

$\mathbb{P}[X_{t_0} = i_0, \dots, X_{t_n} = i_n] =$

$=$