MATH 285: Stochastic Processes

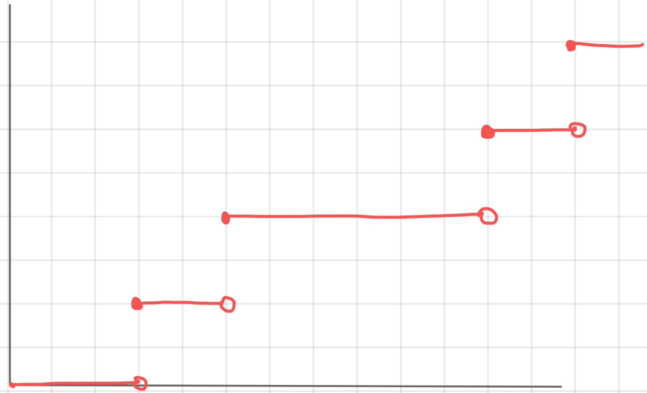
math-old.ucsd.edu/~ynemish/teaching/285

Today: Poisson processes
Birth and death chains
Recurrence and transience

- Homework 5 is due on Friday, March 4, 11:59 PM

Poisson processes

! The jump chain of a Poisson process has a deterministic trajectory



By Prop. 19.2, given the trajectory the sojourn time are independent exponential r.v. with $S_k \sim \mathrm{Exp}(q(Y_{k-1}))$

$$\mathbb{P}[S_1 > s_1, \dots, S_n > s_n] = \sum_{i_0, \dots, i_n} \mathbb{P}[S_1 > s_1, \dots, S_n > s_n \mid Y_0 = i_0, \dots, Y_n = i_n]\, \mathbb{P}[Y_0 = i_0, \dots, Y_n = i_n]$$

$$=$$

$$=$$

<u>Prop 20.6</u> If (X_t) is a Poisson process, then $S_1, S_2, \dots$ are

Poisson processes

Alternative construction of a Poisson process (with $X_0=0$):

- take a collection of i.i.d. random variables S_k, $S_k \sim Exp(\lambda)$
- define the jump times $J_n = S_1 + \cdots + S_n$, $J_0 = 0$
- set $X_t = n$ for $J_n \leq t < J_{n+1}$

Then X_t is a Poisson process with rate λ.

You can think about J_n as the times of some events, and X_t as the number of events that happend up to time t.

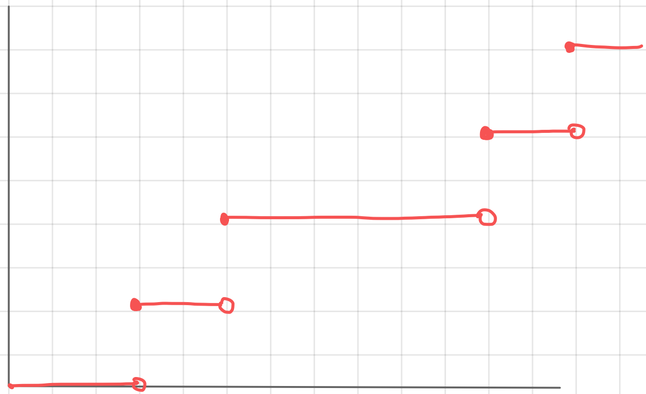
Theorem 20.7 Let $(X_t)_{t\geq 0}$ be a Poisson process of rate λ, $X_0=0$. Then for any $s\geq 0$ the process

is a Poisson process of rate λ, independent of $\{X_u : 0 \leq u \leq s\}$

No proof.

Independent increments

Given a stochastic process $(X_t)_{t \geq 0}$ its increments are random variables



Suppose that (X_t) is a counting process, i.e., (jump times = event times, X_t = # of events that occurred up to time t). Then for $s<t$

$X_t - X_s$ = # of events that occurred on $(s, t]$.

Cor. 20.8 If (X_t) is a Poisson process with rate λ, then for any $0 \leq t_0 < t_1 < \cdots < t_n$ the increments $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$ are independent, and each increment $X_t - X_s$ is a Poisson random variable with rate . These properties uniquely characterize the Poisson process.

Independent increments

Proof. • $X_t - X_s = X_{s+(t-s)} - X_s \sim$

• $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent

Induction : Suppose $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent

By Thm 20.7, for any $t \geq 0$ the process

is independent of X_s for $s \leq t_n$

Therefore, $\tilde{X}_t$ is independent of $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$,

and for any $t_{n+1} > t_n$ $\tilde{X}_{t_{n+1} - t_n} = X_{t_{n+1}} - X_{t_n}$ is independent

of $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$

• Independent increments uniquely determine the joint distribution of $(X_{t_0}, \dots, X_{t_n})$ for any $0 \leq t_0 < \dots < t_n < \infty$

$\mathbb{P}[X_{t_0} = i_0, \dots, X_{t_n} = i_n] =$

$=$

Birth and death chains

Consider a continuous-time MC with state space

$S = \{0, 1, 2, \dots\}$ and transition rates

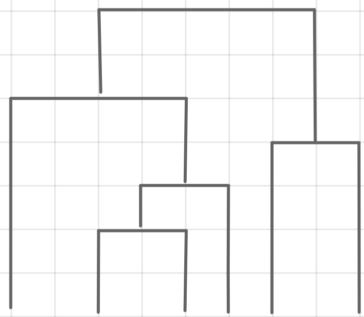
We call this process the birth and death chain.

- all $\mu_i = 0$ pure birth process
- all $\lambda_i = 0$ pure death process
- Poisson process is a pure birth process with $\lambda_i = \lambda$

Example: Kingman's coalescent

Pure death process with $\mu_1 = 0$, $\mu_k = \binom{k}{2}$

Tracking ancestor lines back in time



Kingman's coalescent

$$
\begin{tikzcd}
1 & 2 \arrow[l, bend right, "\binom{2}{1}"'] & 3 \arrow[l, bend right, "\binom{3}{2}"'] & 4 \arrow[l, bend right, "\binom{4}{2}"'] & 5 \arrow[l, bend right, "\binom{5}{2}"'] & 6 \arrow[l, bend right, "\binom{6}{2}"'] & 7 \arrow[l, bend right, "\binom{7}{2}"'] & \cdots
\end{tikzcd}
$$

Denote

the time to most recent common ancestor.

Conditioned on $X_0 = N$, $T = S_1 + S_2 + \cdots + S_{N-1}$, where

S_1 = time spent at state N, S_2 = time spent at $N-1$, ...

$$\mathbb{E}[T] = \mathbb{E}[S_1 + S_2 + \cdots + S_{N-1}] =$$

$$=$$

Denote L = sum of the branch lengths. Compute

Conditioned on $X_0 = N$, $L =$

$$\mathbb{E}[L] =$$

Explosion

Let (X_t) be a pure birth process with $\lambda_i = i^2$.

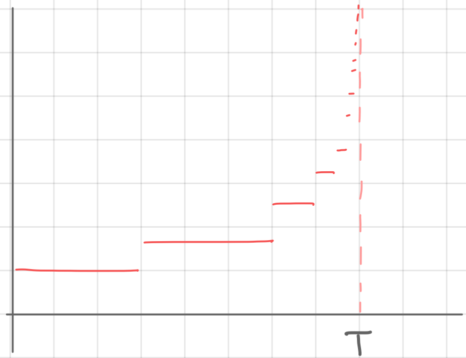
Condition on $X_0 = 1$. Denote by T_N the time to reach N.

Then $T_N = S_1 + S_2 + \cdots + S_{N-1}$ and

$$\mathbb{E}[T_N] =$$

Denote the time to reach infinity. Then

, and thus



We call T the explosion time.

What happens after T?

We can set $X_t = \infty$ for $t \geq T$ (minimal)

or we can restart from another state

Recurrence and transience

<u>Def 21.2</u> Let $(X_t)_{t\geq 0}$ be a continuous-time MC with state space S, and let $i \in S$. Let $T_i = \min\{t > 0 : X_t = i\}$.

The state i is called transient if $\mathbb{P}_i[T_i < \infty] = 0$.

recurrent if $\mathbb{P}_i[T_i < \infty] = 1$

positive recurrent if $\mathbb{E}_i[T_i] < \infty$

- i is recurrent (transient) for (X_t) iff i is recurrent (transient) for the embedded jump chain (Y_n)

  X_t revisits i infinitely many times

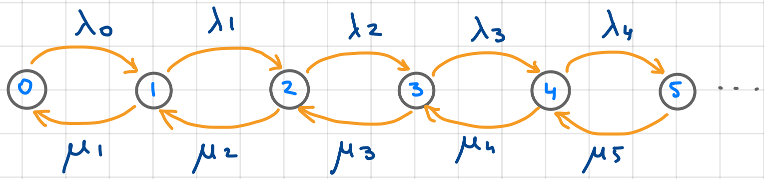
  iff Y_n revisits i infinitely many times

- Positive recurrence takes into account how long it takes to revisit i

Recurrence for birth and death chains

Let $(X_t)_{t\geq 0}$ be a birth and death chain with parameters

$\lambda_i = q(i,i+1) > 0$ for $i \geq 0$, $\mu_i = q(i,i-1) > 0$. for $i \geq 1$



(X_t) is irreducible (all $\lambda_i > 0$, $\mu_i > 0$), so it is enough to analyze one state for recurrence/transience (take state 0).

Similarly as for the discrete-time MC, denote

$$h(i) :=$$

Then

FSA $\Bigg\{$

Recurrence for birth and death chains

By the Strong Markov property

$$h(i) = \qquad\qquad (*)$$

Recall that $p(i,j) = q(i,j)/q(i)$, so $(*)$ becomes

$$h(i) =$$

We can rewrite this using the differences

$$h(i+1) - h(i) =$$

Applying the above identities recursively gives

$$h(i+1) - h(i) =$$

$$=$$

Recurrence for birth and death chains

After taking the partial sums

$$h(n) - h(0) =$$

- if $\sum_{i=0}^{\infty} \rho_i = \infty$, then , and $\forall n \geq 1$

  ↳ (X_t) is recurrent

- if $\sum_{i=0}^{\infty} \rho_i < \infty$, we need to find the minimal solution (Thm 7.0)

  which is achieved when $h(0) - h(1) =$

Then $h(1) = 1 - \dfrac{1}{\sum_{i=0}^{\infty} \rho_i} < 1$ and (X_t) is transient.