MATH 285: Stochastic Processes

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Today: Martingales

- Homework 6 is due on Friday, March 4, 11:59 PM

Martingales

Def 24.1 A discrete-time martingale is a stochastic process $(X_n)_{n\geq 0}$ which satisfies $\mathbb{E}[|X_n|]<\infty$ and

$$\mathbb{E}[X_{n+1} \mid X_0,\dots,X_n] = X_n \quad \text{for all } n\geq 0$$

Lemma 24.2 If $(X_n)_{n\geq 0}$ is a martingale, then

$$\mathbb{E}[X_n \mid X_0,\dots,X_m] = X_m \quad \text{for all } m<n.$$

Proof. Fix m. Induction. Holds for $n=m$, $n=m+1$.

Suppose $\mathbb{E}[X_n \mid X_0,\dots,X_m] = X_m$ for some $n>m$. Then by the Tower property

$$\mathbb{E}[X_{n+1} \mid X_0,\dots,X_m] = \mathbb{E}[\,\mathbb{E}[X_{n+1} \mid X_0,\dots,X_n] \mid X_0,\dots,X_m]$$

$$= \mathbb{E}[X_n \mid X_0,\dots,X_m] = X_m$$

$\blacksquare$

Martingales

Corollary 24.3 If $(X_n)_{n\geq 0}$ is a martingale, then it has constant expectation: $\mathbb{E}[X_n] = \mathbb{E}[X_0]$ for all n.

Proof. Use the double expectation property

$$\mathbb{E}[X_n] = \mathbb{E}\left[\mathbb{E}[X_n \mid X_0]\right] = \mathbb{E}[X_0] \quad \blacksquare$$

Example (Betting on independent coin tosses)

Consider a game: bet B_i dollars and toss a coin.

$$X_i = \begin{cases} 1, & \text{if you win the } i\text{-th toss} \\ -1, & \text{if you lose the } i\text{-th toss} \end{cases}, \quad X_1, X_2, \dots \text{ independent}$$

Denote by W_0 the initial fortune, independent of $X_1, X_2, \dots$

Let $W_n = W_0 + \sum_{i=1}^{n} X_i B_i$. We call $B_1, B_2, \dots$ the betting strategy.

- $\mathbb{E}[|B_n|] < \infty$
- B_n is $(W_0, \dots, W_{n-1})$-measurable

Betting on independent coin tosses

Then

$$\mathbb{E}[|W_n|] \le \mathbb{E}[|W_0| + \sum_{i=1}^{n} |X_i B_i|] = \mathbb{E}(|W_0|) + \sum_{i=1}^{n} \mathbb{E}[|B_i|] < \infty \quad \forall n$$

and

$$\mathbb{E}[W_{n+1} | W_0, \dots, W_n] = \mathbb{E}[W_n + B_{n+1} X_{n+1} | W_0, \dots, W_n]$$

$$= W_n + \mathbb{E}[X_{n+1} B_{n+1} | W_0, \dots, W_n]$$

$$= W_n + B_{n+1} \mathbb{E}[X_{n+1} | W_0, \dots, W_n]$$

Since $W_k = \sum_{i=1}^{k} X_i B_i$, X_{n+1} is independent of $W_0, \dots, W_n$

W_0 is independent of X_{n+1}

B_1 is W_0 measurable, $W_1 = W_0 + X_1 B_1$

B_2 is (W_0, W_1) measurable, $W_2 = W_1 + X_2 B_2$ ...

Then $\mathbb{E}[X_{n+1} | W_0, \dots, W_n] = \mathbb{E}[X_{n+1}] = 0$, and $\mathbb{E}[W_{n+1} | W_0, \dots, W_n] = W_n$

i.e., W_n is a martingale

Stopping times

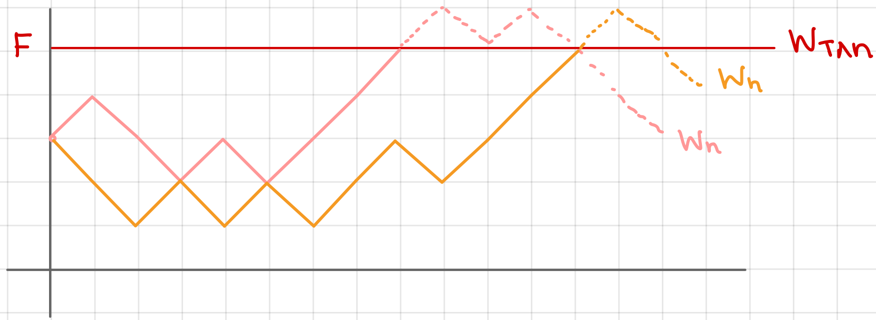
Let $(W_n)_{n\geq 0}$ be a stochastic process. Recall that random variable $T \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ is a stopping time if the fact that $\{T \leq n\}$ holds can be determined from $W_0, \ldots, W_n$

Example of a stopping times: the first time the process hits some set/value, $T = \min\{n \geq 0 : W_n \geq F\}$

Suppose you stop the game as soon as your fortune gets $\geq F$.
Then the original process (W_n) is replaced by $(W_{T\wedge n})_{n\geq 0}$,

where $T \wedge n = \min\{T, n\}$

$$W_{T\wedge n} = \begin{cases} W_n, & \text{if } n < T \\ W_T, & \text{if } n \geq T \end{cases}$$



Stopped martingale

Prop. 24.5 Let $(X_n)_{n\geq 0}$ be a martingale and let T be a stopping time for this martingale. Then

$$(X_{T\wedge n})_{n\geq 0} \quad \text{is a martingale.}$$

Proof. Denote $Y_n := X_{T\wedge n}$. Then $Y_n = X_{T\wedge n} \in \{X_0, X_1, \dots, X_n\}$ for each n, so $|Y_n| \leq \max\{|X_0|, \dots, |X_n|\} \leq |X_0| + \dots + |X_n|$, and

$$\mathbb{E}[|Y_n|] \leq \mathbb{E}[|X_0| + \dots + |X_n|] < \infty$$

Now we need to show that $\mathbb{E}[Y_{n+1} \mid Y_0, \dots, Y_n] = Y_n$.

(1) $\mathbb{E}[Y_{n+1} \mid X_0, \dots, X_n] = Y_n$

- $Y_{n+1} = X_{T\wedge(n+1)} = X_T \mathbb{1}_{\{T\leq n\}} + X_{n+1} \mathbb{1}_{\{T>n\}}$
- $\{T\leq n\}$ only depends on $X_0, \dots, X_n$, so $\mathbb{1}_{\{T\leq n\}}$ is $(X_0, \dots, X_n)$-measurable and $X_T \mathbb{1}_{\{T\leq n\}}$ is $(X_0, \dots, X_n)$ measurable

Stopped martingale

- $\mathbb{1}_{\{T>n\}} = 1 - \mathbb{1}_{\{T\le n\}}$ is $(X_0,\dots,X_n)$-measurable
- Using the properties of the conditional expectation

$$\begin{aligned}\mathbb{E}[Y_{n+1} \mid X_0,\dots,X_n] &= \mathbb{E}[X_T\,\mathbb{1}_{\{T\le n\}} \mid X_0,\dots,X_n] + \mathbb{E}[X_{n+1}\,\mathbb{1}_{\{T>n\}} \mid X_0,\dots,X_n] \\ &= X_T\,\mathbb{1}_{\{T\le n\}} + \mathbb{1}_{\{T>n\}}\,\mathbb{E}[X_{n+1} \mid X_0,\dots,X_n] \\ &= X_T\,\mathbb{1}_{\{T\le n\}} + \mathbb{1}_{\{T>n\}}\,X_n\end{aligned}$$

- $X_T\,\mathbb{1}_{\{T\le n\}} + X_n\,\mathbb{1}_{\{T>n\}} = X_T\,\mathbb{1}_{\{T<n\}} + \mathbb{1}_{\{T\ge n\}}\,X_n = Y_n$

(2) $\mathbb{E}[Y_{n+1} \mid Y_0,\dots,Y_n] = Y_n$

- $\bar{Y}_n := (Y_0,\dots,Y_n)$ is $\bar{X}_n := (X_0,\dots,X_n)$ measurable
- Using the Tower property $\mathbb{E}[Y_{n+1} \mid \bar{Y}_n] = \mathbb{E}[\mathbb{E}[Y_{n+1} \mid \bar{X}_n] \mid \bar{Y}_n]$
- By (1) $\mathbb{E}[\mathbb{E}[Y_{n+1} \mid \bar{X}_n] \mid \bar{Y}_n] \overset{(1)}{=} \mathbb{E}[Y_n \mid \bar{Y}_n] = Y_n$ $\blacksquare$

Martingale betting strategy

Corollary For any $n \in \mathbb{N}$ $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_{T\wedge 0}] = \mathbb{E}[X_0]$

$\mathbb{E}[X_T]$

Example (Martingale betting strategy)

Consider the betting strategy $B_n = 2^{n-1}$ (double each round).

Let $T = \min\{n : X_n = 1\}$, the time of the first win. If $W_0 = C$,

then

$$\mathbb{E}[W_T] = \sum_{n=1}^{\infty} \mathbb{E}[W_T \mid T = n]\, \mathbb{P}[T = n]$$

The event $T = n$ corresponds to a specific trajectory

$X_1 = -1, X_2 = -1, \dots, X_{n-1} = -1, X_n = 1$, so $\mathbb{P}[T=n] = \frac{1}{2^n}$

So $$\mathbb{E}[W_T] = \sum_{n=1}^{\infty} \mathbb{E}[W_n \mid X_1 = -1, \dots, X_{n-1} = -1, X_n = 1]\, \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} \left(C - 1 - 2 - 4 \cdots - 2^{n-2} + 2^{n-1}\right) \frac{1}{2^n} = C + 1 \neq C = \mathbb{E}[W_0]$$

Problem T can be arbitrarily large.

Optional Sampling Theorem

Thm 24.8 Let $(X_n)_{n\geq 0}$ be a martingale, and let T be a finite stopping time. Suppose that either

(1) T is bounded : $\exists\, N < \infty$ s.t. $\mathbb{P}[T < N] = 1$; or

(2) $(X_n)_{0 \leq n \leq T}$ is bounded : $\exists\, B < \infty$ s.t. $\mathbb{P}[\,|X_n| \leq B$ for all $n \leq T] = 1$

Then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$

Proof.

- Suppose (1) holds. By Prop. 24.5 $X_{T\wedge n}$ is a martingale, $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_{T\wedge 0}] = \mathbb{E}[X_0]$ for all n.

  Then $\mathbb{E}[X_0] = \mathbb{E}[X_{T\wedge N}] = \mathbb{E}[X_T]$

- Suppose that (2) holds (T is not necessarily bounded)

  Then $X_T = X_{T\wedge n} + (X_T - X_{T\wedge n}) = X_{T\wedge n} + (X_T - X_{T\wedge n})\mathbb{1}_{\{T > n\}}$

  First term: $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_0]$

Optional Sampling Theorem

Second term:

$$\left|\mathbb{E}\left[(X_T - X_{T\wedge n})\mathbb{1}_{\{T>n\}}\right]\right| \leq \mathbb{E}\left[\,|X_T - X_{T\wedge n}|\,\mathbb{1}_{\{T>n\}}\right]$$

$$\leq 2B\,\mathbb{P}[T>n]$$

Since $\mathbb{P}[T<\infty]=1$, $\lim\limits_{n\to\infty} \mathbb{P}[T>n] = 0$

Therefore,

$$\left|\mathbb{E}[X_T] - \mathbb{E}[X_0]\right| = \left|\mathbb{E}\left[(X_T - X_{T\wedge n})\mathbb{1}_{\{T>n\}}\right]\right| \leq \underbrace{2B\,\mathbb{P}[T>n]}_{\xrightarrow{n\to\infty} 0}$$

$\blacksquare$