MATH 10C: Calculus III (Lecture B00)

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Today: Calculus of vector-valued functions. Tangent lines

Next: Strang 3.4

Week 4:

- homework 4 (due Friday, October 27)

Derivatives of vector-valued functions

The derivative of a vector-valued function $\vec{r}$ is

$$\frac{d\vec{r}(t)}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

provided that the limit exists. If $\vec{r}\,'(t)$ exists, we say that $\vec{r}$ is differentiable at t. If $\vec{r}$ is differentiable at every point t from the interval (a,b), we say that $\vec{r}$ is differentiable on (a,b).

Notice that if $\vec{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$, then

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\langle r_1(t+h) - r_1(t),\ r_2(t+h) - r_2(t),\ r_3(t+h) - r_3(t) \rangle}{h}$$

$$= \left\langle \lim_{h \to 0} \frac{r_1(t+h) - r_1(t)}{h},\ \lim_{h \to 0} \frac{r_2(t+h) - r_2(t)}{h},\ \lim_{h \to 0} \frac{r_3(t+h) - r_3(t)}{h} \right\rangle = \langle r_1'(t), r_2'(t), r_3'(t) \rangle$$

Calculus of vector-valued functions

Example Let $\vec{r}(t) = \langle \sin t,\ e^{2t},\ t^2 - 4t + 2 \rangle$

Then $\vec{r}\,'(t) = \langle \cos t,\ 2e^{2t},\ 2t - 4 \rangle$

Summary

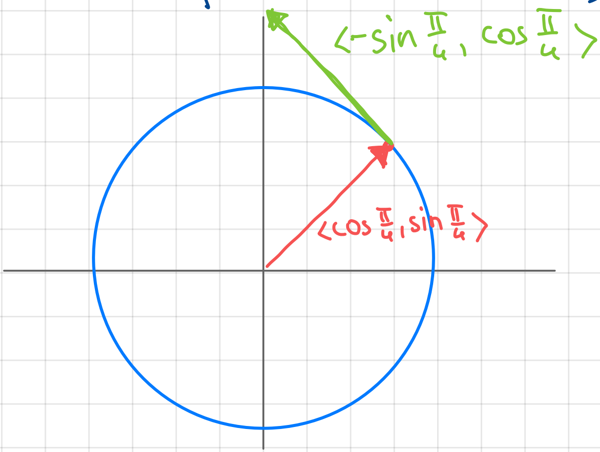
Calculus concepts (limit, continuity, derivative) are applied to vector-valued functions componentwise (apply to each component separately).

If $\vec{r}(t)$ represents the position of some object, then

- $\vec{r}\,'(t)$ is the velocity of this object ($\|\vec{r}\,'(t)\|$ is speed)
- $\vec{r}\,''(t)$ is the acceleration of the object

Tangent vectors. Tangent lines

Let $\vec{r}(t)$ be a vector-valued function. Suppose that $\vec{r}$ is differentiable at t_0. Let C be a curve defined (parametrized) by $\vec{r}(t)$



$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{r}\,'\left(\frac{\pi}{4}\right) = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

Then vector $\vec{r}\,'(t)$ is tangent to C at t_0 (at $\vec{r}(t_0)$)

The tangent line to $\vec{r}$ at t_0 is the line given by the vector equation $\vec{\ell}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)(t - t_0)$

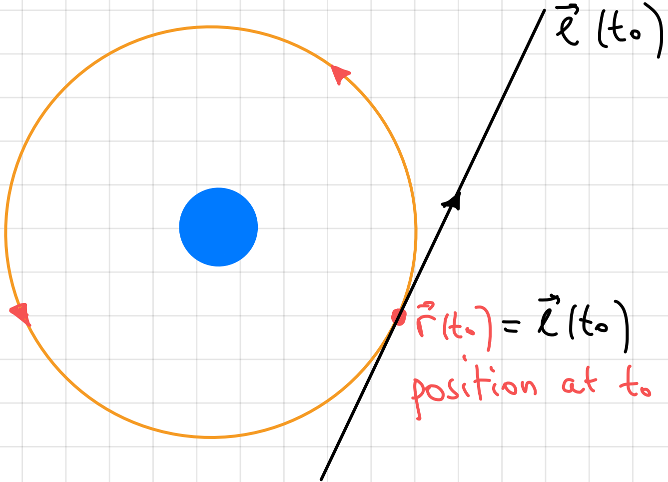
Tangent vectors. Tangent lines

The tangent line $\vec{\ell}(t)$ to $\vec{r}(t)$ at t_0 has the same position and velocity as $\vec{r}$ at time t_0:

$$\vec{\ell}(t_0) = \vec{r}(t_0)$$

$$\vec{\ell}\,'(t_0) = \vec{r}\,'(t_0)$$

Example Imagine satellite orbiting a planet.



If the planet disappears at time t_0, then the satellite would keep going along $\vec{\ell}(t)$

Tangent vectors. Tangent lines

Example Let $\vec{r}(t) = \langle t^2 - 2, e^{3t}, t \rangle$

Find the tangent line to $\vec{r}(t)$ at $t_0 = 1$.

First, find the tangent vector at $t_0 = 1$

$$\vec{r}\,'(t) = \langle 2t, 3e^{3t}, 1 \rangle$$

$$\vec{r}\,'(1) = \langle 2, 3e^3, 1 \rangle$$

Next, find the position at $t_0 = 1$

$$\vec{r}(1) = \langle -1, e^3, 1 \rangle$$

Finally, we can write the equation for the tangent line

$$\vec{\ell}(t) = \langle -1, e^3, 1 \rangle + \langle 2, 3e^3, 1 \rangle \cdot (t-1)$$

Definition We call $T(t) := \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ the principal unit tangent vector to $\vec{r}$ at t. (provided $\|\vec{r}\,'(t)\| \neq 0$)

Integrals of vector-valued functions

Integration of vector-valued functions is done componentwise: if $\vec{r}(t) = \langle r_1(t), r_2(t), r_3(t)\rangle$, then

$$\int \vec{r}(t)\,dt = \left\langle \int r_1(t)\,dt, \int r_2(t)\,dt, \int r_3(t)\,dt \right\rangle \text{ (antiderivative)}$$

and if $a < b$

$$\int_a^b \vec{r}(t)\,d = \left\langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \int_a^b r_3(t)\,dt \right\rangle \text{ (definite integral)}$$

Example

- $\int \langle \sin t, t^2+2t, e^{2t} \rangle\,dt = \left\langle -\cos t + c_1, \frac{t^3}{3} + t^2 + c_2, \frac{e^{2t}}{2} + c_3 \right\rangle$

$$= \left\langle -\cos t, \frac{t^3}{3} + t^2, \frac{e^{2t}}{2} \right\rangle + \vec{c},$$ where $\vec{c}$ is an arbitrary vector of constants

- $\int_0^2 \left(\sin t \cdot \vec{i} + (t^2+2t)\cdot\vec{j} + e^{2t}\vec{k}\right)dt = \left\langle -\cos(2) + \cos(0), \frac{2^3}{3} + 2^2 - 0, \frac{e^{2\cdot 2}}{2} - \frac{e^{2\cdot 0}}{2} \right\rangle$

$$= \left\langle -\cos(2) + 1, \frac{8}{3} + 4, \frac{e^4}{2} - \frac{1}{2} \right\rangle$$

Integrals of vector-valued functions

Fundamental theorem of calculus

Let $\vec{f}: [a,b] \to \mathbb{R}^3$ be a continuous vector-valued function.
Let $\vec{F}: [a,b] \to \mathbb{R}^3$ be such that $\vec{F}' = \vec{f}$ ($\vec{F}$ is antiderivative of $\vec{f}$). Then $\int_a^b \vec{f}(t)\,dt = \vec{F}(b) - \vec{F}(a)$

In particular,

- if $\vec{v}(t)$ is the velocity vector, $\vec{r}(t)$ is the position, then
$$\int_a^b \vec{v}(t)\,dt = \vec{r}(b) - \vec{r}(a)$$
gives the displacement between times a and b
- if $\vec{a}(t)$ is the acceleration, then $\int_a^b \vec{a}(t)\,dt = \vec{v}(b) - \vec{v}(a)$

Properties of derivatives of vector-valued functions

Thm 3.3. Let $\vec{r}(t)$ and $\vec{u}(t)$ be differentiable vector-valued functions, let $f(t)$ be a differentiable scalar function, let c be a scalar.

(i) $\frac{d}{dt}[c\vec{r}(t)] = c\,\vec{r}\,'(t)$ (scalar multiple)

(ii) $\frac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] = \vec{r}\,'(t) \pm \vec{u}\,'(t)$ (sum and difference)

(iii) $\frac{d}{dt}[f(t)\,\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)\cdot\vec{r}\,'(t)$ (product with scalar function)

(iv) $\frac{d}{dt}[\vec{r}(t)\cdot\vec{u}(t)] = \vec{r}\,'\cdot\vec{u} + \vec{r}\cdot\vec{u}\,'$ (dot product)

(v) $\frac{d}{dt}[\vec{r}(t)\times\vec{u}(t)] = \vec{r}\,'(t)\times\vec{u}(t) + \vec{r}(t)\times\vec{u}\,'(t)$ (cross product)

(vi) $\frac{d}{dt}[\vec{r}(f(t))] = \vec{r}\,'(f(t))\cdot f'(t)$ (chain rule)

Properties of derivatives of vector-valued functions

(vii) If $\vec{r}(t) \cdot \vec{r}(t) = c$, then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$

Proof (iv) $\dfrac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)]$

$=$

$=$

$=$

(vii) $\dfrac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)]$

This means that if $\|\vec{r}(t)\|$ is constant, then