MATH 10C: Calculus III (Lecture B00)

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Today: Projectile motion. Functions of two variables

Next: Strang 4.1

Week 5:

- homework 4 (due Friday, October 27)
- regrades of Midterm 1 on Gradescope until October 29

Properties of derivatives of vector-valued functions

Thm 3.3. Let $\vec{r}(t)$ and $\vec{u}(t)$ be differentiable vector-valued functions, let $f(t)$ be a differentiable scalar function, let c be a scalar.

(i) $\frac{d}{dt}[c\vec{r}(t)] =$ (scalar multiple)

(ii) $\frac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] =$ (sum and difference)

(iii) $\frac{d}{dt}[f(t)\vec{r}(t)] =$ (product with scalar function)

(iv) $\frac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)] =$ (dot product)

(v) $\frac{d}{dt}[\vec{r}(t) \times \vec{u}(t)] =$ (cross product)

(vi) $\frac{d}{dt}[\vec{r}(f(t))] =$ (chain rule)

Properties of derivatives of vector-valued functions

(vii) If $\vec{r}(t) \cdot \vec{r}(t) = c$, then

Proof (iv) $\dfrac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)]$

$=$

$=$

$=$

(vii) $\dfrac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)]$

This means that if $\|\vec{r}(t)\|$ is constant, then

Motion in space

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the position of the particle at time t, then

- $\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ is the velocity, and
- $\vec{a}(t) = \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle$ is the acceleration, and
- $v(t) = \|\vec{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ is the speed

Example: Projectile motion

Consider an object moving with initial velocity $\vec{v}_0$ but with no forces acting on it other than gravity (ignore the effect of air resistance).

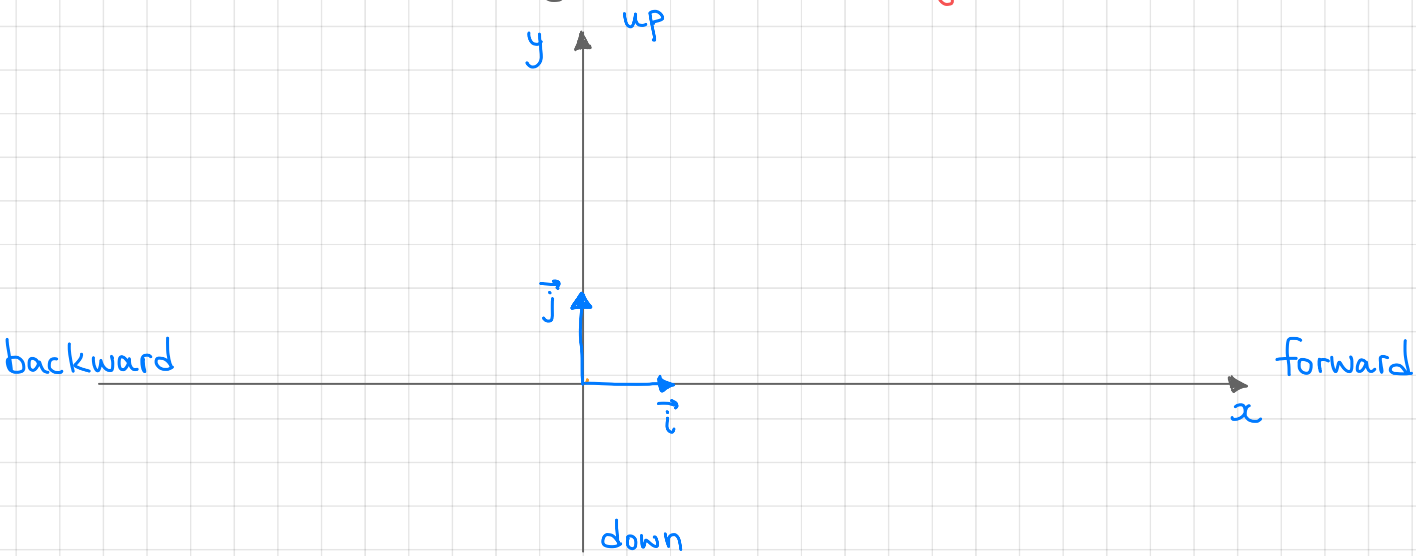
Newton's second law: , where m = mass of the object

Earth's gravity: , where $g \approx 9.8\ m/s^2$

Projectile motion

Fix the coordinate system:

$\vec{F}_g =$



Newton's second law: $\vec{F} = m\vec{a}$

Earth's gravity: $\vec{F}_g =$

Earth's gravity is the only force acting on the object

Projectile motion

$$\vec{F}(t) = \vec{F}_g :$$

(constant acceleration)

Since $\vec{a}(t) = \vec{v}'(t)$, we have

Take antiderivative: $\vec{v}(t) =$

Determine $\vec{c}_1$ by taking (initial velocity):

$$\vec{v}(0) =$$

This gives the velocity of the object:

$$\vec{v}(t) =$$

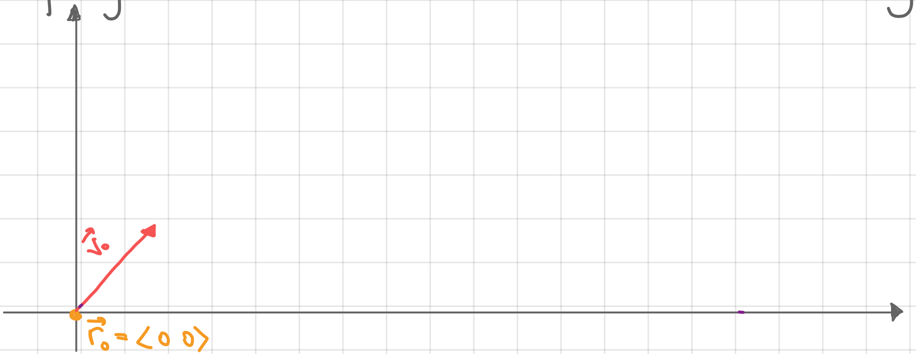
Similarly, $\vec{v}(t) =$. By taking the antiderivative and $\vec{r}(0) = \vec{r}_0$

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{c}_0 =$$

$$\vec{r}(0) = \vec{c}_0 = \vec{r}_0, \quad \text{so}$$

Projectile motion

A projectile is shot by a howitzer with initial speed 800 m/s on a flat terrain. Determine the max distance the projectile can cover before hitting the ground.



Since the initial speed is given, the initial velocity can be determined by the angle :

Equation of the trajectory: $\vec{r}(t) =$

Hitting the ground: second component or $\vec{r}(t)$ is 0:

, so . The position of the hit is

$\vec{r}(t_h) =$. Maximum is achieved when , i.e., . Max distance is 32 km.

Functions of several variables

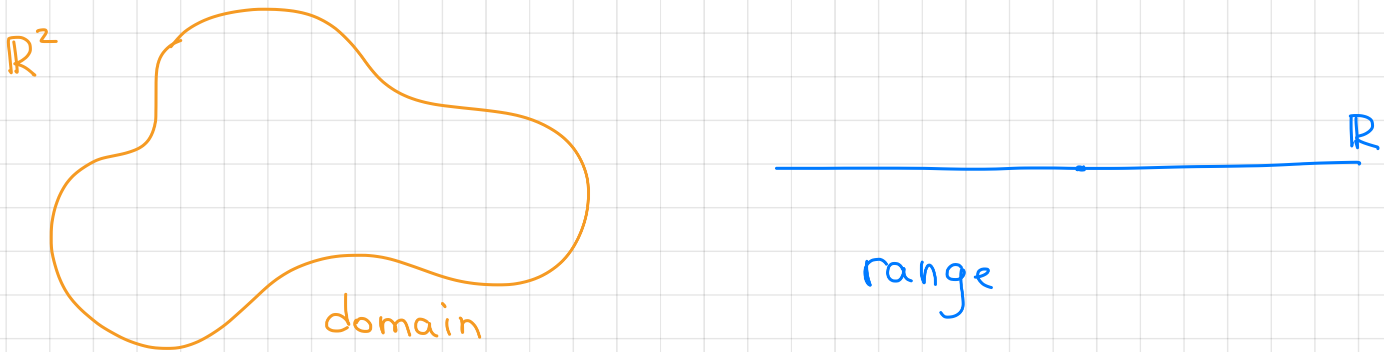
Functions of two variables

Def. A function of two variables maps each

in a subset $D \subset \mathbb{R}^2$ to a

. The set D is called the of the

function. The range of f is the set of all real numbers z that has at least one ordered pair $(x,y) \in D$ s.t. $f(x,y) = z$.



If not specified, we choose the domain to be the set of all pairs (x,y) for which $f(x,y)$ is well-defined.

Functions of two variables

Example Find the domain and range of the function

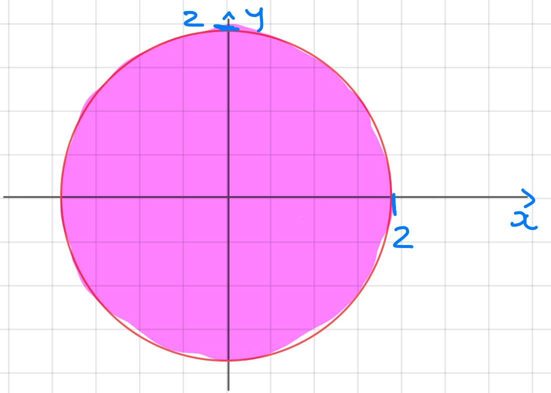
$$f(x,y) = \sqrt{4-x^2-y^2}$$

Domain. One restriction: the number under the square roof has to be nonnegative, i.e.,

The set of all pairs $(x,y) \in \mathbb{R}^2$ such that

is a

The domain of f is



Range. For (x,y) in the domain

the range of x^2+y^2 is

the range of $4-x^2-y^2$ is

the range of $\sqrt{4-x^2-y^2}$ is

Graph of a function of two variables

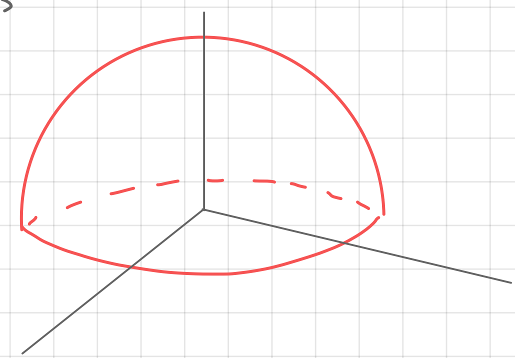
Function f of two variables: maps each pair (x,y) from its domain to a real number $z=f(x,y)$.

The graph of f consists of ordered triples $(x,y,f(x,y))$ for all (x,y) in the domain of f. We call the graph of a function of two variables a surface.

Example $f(x,y)=\sqrt{4-x^2-y^2}$, $\mathrm{dom}(f)=\{(x,y)\mid x^2+y^2\le 4\}$

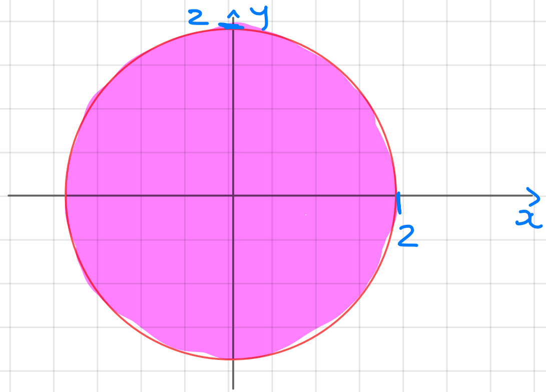
Graph of f consists of all $(x,y,z)\in\mathbb{R}^3$ such that $z=\sqrt{4-x^2-y^2}$, or

- equation of a



Level curves

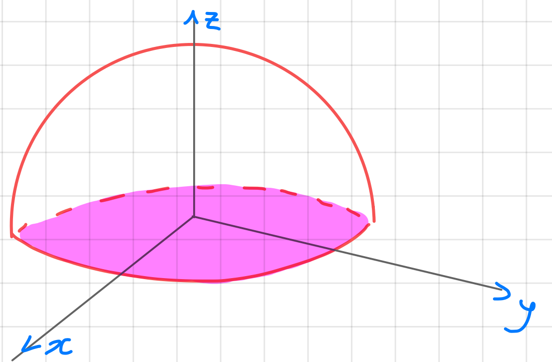
Def. Given a function $f(x,y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be



Example $f(x,y) = \sqrt{4 - x^2 - y^2}$

Range of f is $[0, 2]$.

Take . Then the level curve of f for is defined by



Functions of more than two variables

In a similar way we can define functions of more than two variables, e.g., functions of three variables:

to each point (x, y, z) in the domain assign a real number $f(x, y, z)$.

<u>Example</u> $f(x, y, z) =$; domain: all points $(x, y, z) \in \mathbb{R}$

such that , i.e.