MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Functions of two variables

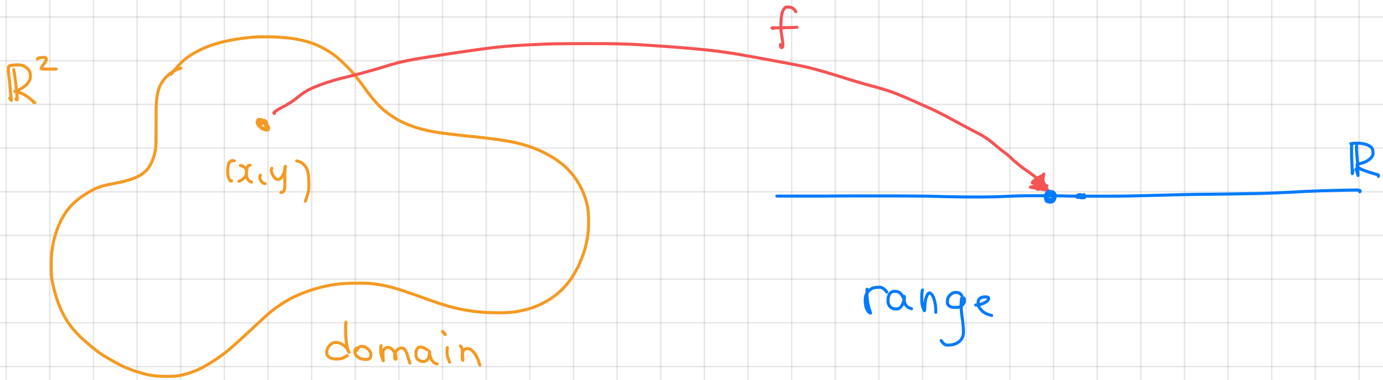
Next: Strang 4.2

Week 5:

- homework 4 (due Friday, October 27)
- regrades of Midterm 1 on Gradescope until October 29

Functions of two variables

Def. A function of two variables maps each ordered pair (x,y) in a subset $D \subset \mathbb{R}^2$ to a unique real number $z = f(x,y)$. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x,y) \in D$ s.t. $f(x,y) = z$.



If not specified, we choose the domain to be the set of all pairs (x,y) for which $f(x,y)$ is well-defined.

Functions of two variables

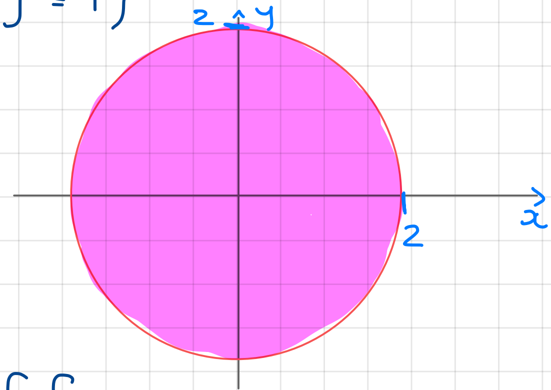
Example Find the domain and range of the function

$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

Domain. One restriction: the number under the square root has to be nonnegative, i.e., $4 - x^2 - y^2 \geq 0$

The set of all pairs $(x,y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 4$ is a disk of radius 2 centered at the origin

The domain of f is $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$



Range. For (x,y) in the domain

the range of $x^2 + y^2$ is interval $[0,4]$

the range of $4 - x^2 - y^2$ is interval $[0,4]$

the range of $\sqrt{4 - x^2 - y^2}$ is interval $[0,2]$

range of f

Graph of a function of two variables

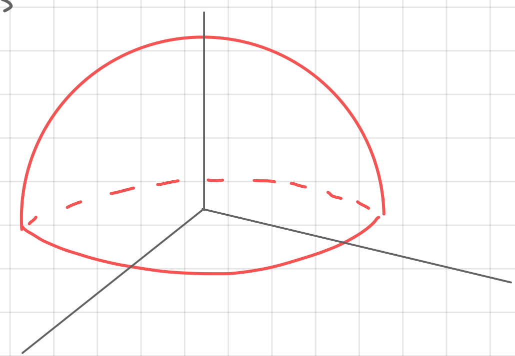
Function f of two variables: maps each pair (x,y) from its domain to a real number $z=f(x,y)$.

The graph of f consists of ordered triples $(x,y,f(x,y))$ for all (x,y) in the domain of f. We call the graph of a function of two variables a surface.

Example $f(x,y)=\sqrt{4-x^2-y^2}$, $\operatorname{dom}(f)=\{(x,y)\mid x^2+y^2\le 4\}$

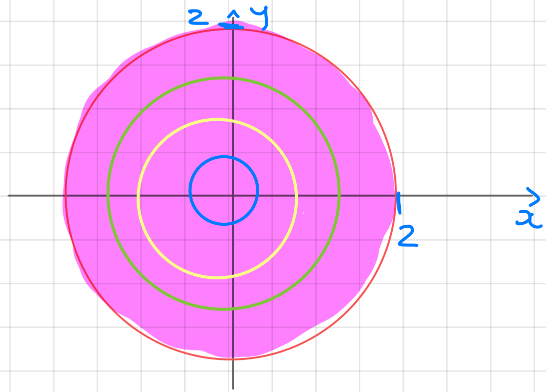
Graph of f consists of all $(x,y,z)\in\mathbb{R}^3$ such that $z=\sqrt{4-x^2-y^2}$, or

$x^2+y^2+z^2=4$ – equation of a sphere of radius 2 centered at $(0,0,0)$ (only the top half)



Level curves

Def. Given a function $f(x,y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of all points (x,y) satisfying the equation $f(x,y)=c$



Example $f(x,y)=\sqrt{4-x^2-y^2}$

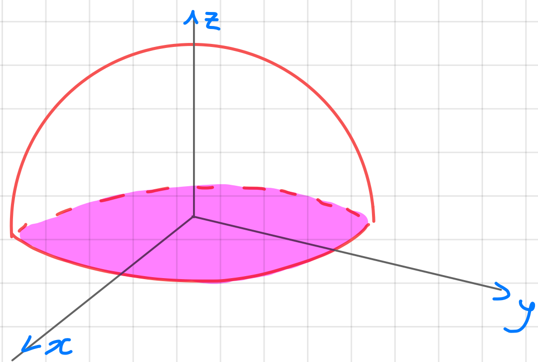
Range of f is $[0,2]$.

Take $c=1$. Then the level curve of f for $c=1$ is defined by $f(x,y)=1$

$$\sqrt{4-x^2-y^2}=1$$

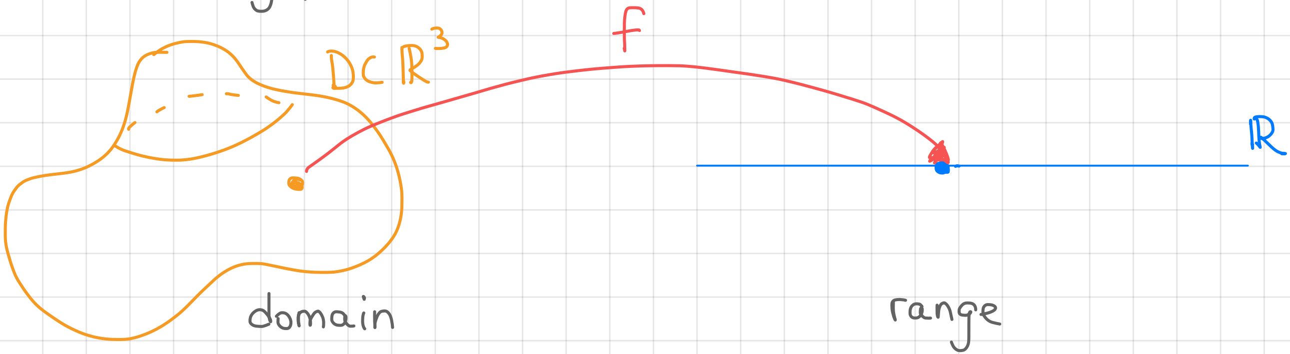
$$x^2+y^2=3$$

circle of radius $\sqrt{3}$ centered at $(0,0)$



Functions of more than two variables

In a similar way we can define functions of more than two variables, e.g., functions of three variables:



to each point (x, y, z) in the domain assign a real number $f(x, y, z)$.

Example $f(x,y,z) = \dfrac{1}{\sqrt{4-x^2-y^2-z^2}}$; domain: all points $(x,y,z) \in \mathbb{R}$ such that $4-x^2-y^2-z^2 > 0$, i.e. the ball of radius 2 centered at $(0,0,0)$ without the boundary/surface, $\operatorname{dom}(f) = \{(x,y,z) \mid x^2+y^2+z^2 < 4\}$

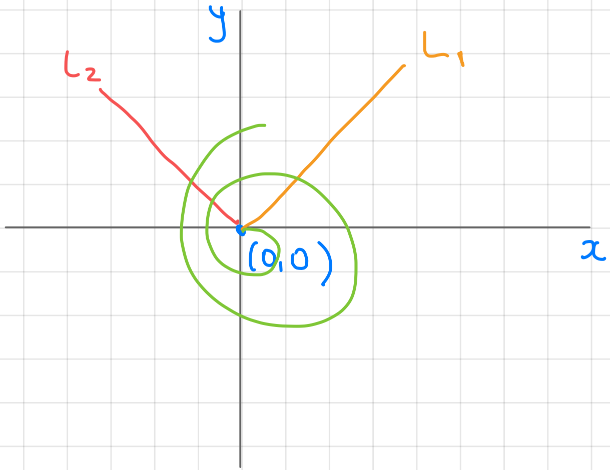
Limit of a function of two variables

<u>Informally</u> $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ means that $f(x,y)$ gets closer to L as (x,y) gets closer to (x_0,y_0).

We have to be careful with what "(x,y) gets closer to (x_0,y_0)" means!

For example,

let $f(x,y) = \frac{x}{y}$



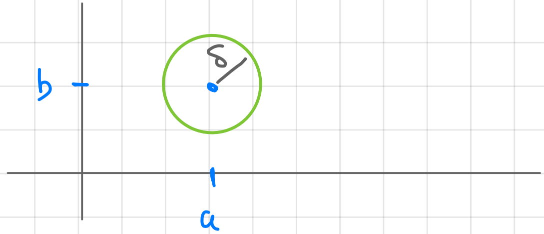
- If (x,y) approaches $(0,0)$ along the line $y=x$ $\lim_{L_1 \ni (x,y)\to(0,0)} \frac{x}{y} = 1$

- If (x,y) approaches $(0,0)$ along the line $y=-x$ $\lim_{L_2 \ni (x,y)\to(0,0)} \frac{x}{y} = -1$

Limit of a function of two variables

<u>Def</u> Consider a point $(a,b) \in \mathbb{R}^2$. A δ-disk centered at point (a,b) is the open disk of radius δ centered at (a,b)

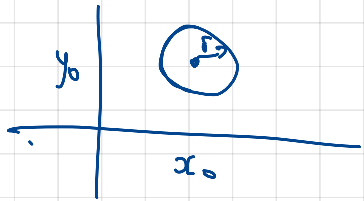
$$\{(x,y) \mid (x-a)^2 + (y-b)^2 < \delta^2\}$$



<u>Def</u>. The limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that all points in a δ-disk around (x_0,y_0), except possible (x_0,y_0) itself, $f(x,y)$ is no more than ε away from L.



For any $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$