MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Local minima/maxima

Next: Strang 4.7

Week 8:

- Midterm 2: Wednesday, November 16 (lectures 10-19)

Last time

Def. Let $z=f(x,y)$ be a function of two variables defined at (x_0,y_0). Then (x_0,y_0) is called a critical point of f if either

- $f_x(x_0,y_0)=0$, $f_y(x_0,y_0)=0$ (i.e. $\nabla f(x_0,y_0)=\vec{0}$); or
- $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ does not exist

Last time

Def Let $z = f(x,y)$ be a function of two variables.
Then f has a local maximum at point (x_0, y_0) if

(*) $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local maximum value. If (*) holds for all (x,y) in the domain of f, we say that f has global maximum at (x_0, y_0).

Function f has a local minimum at point (x_0, y_0) if

(**) $f(x_0, y_0) \leq f(x,y)$ for all points (x,y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called the local minimum value. If (**) holds for all (x,y) in the domain of f, we say that f has global minimum at (x_0, y_0).

Local minima and local maxima are called local extrema.

Last time

<u>Thm 4.16</u> Let $z = f(x,y)$ be a function of two variables. Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f (i.e. $\nabla f(x_0, y_0) = 0$).

<u>Example</u> At the very top of a mountain the ground is flat. If there was slope in some direction, then you could go higher. Similarly, at the lowest point of a crater the ground is also flat ($\nabla f = 0$).

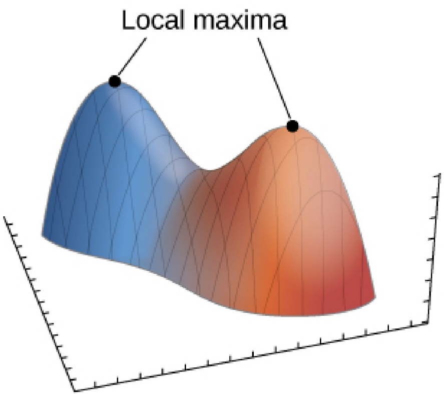
But the fact that the ground is flat ($\nabla f(x_0, y_0) = 0$) does not necessarily mean that f has a local extremum at (x_0, y_0), it may be a saddle point.

Saddle points

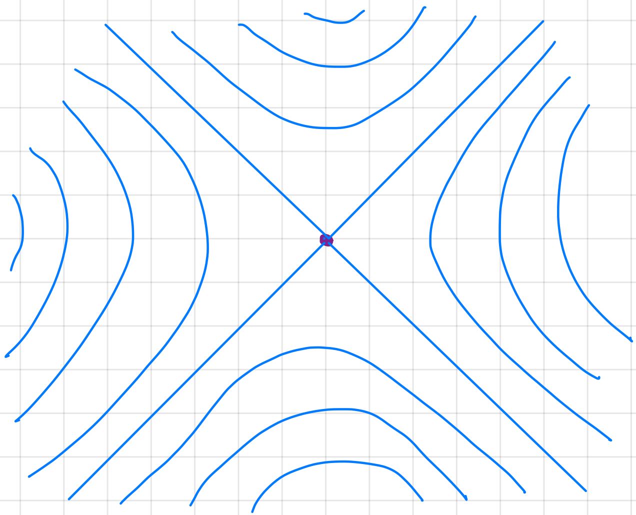
<u>Def.</u> Let $z = f(x,y)$ be a function of two variables.

We say that (x_0, y_0) is a if ,

, but f



Level curves around the saddle point have this shape



The second derivative test

Thm 4.17 (Second derivative test)

Suppose that $f(x,y)$ is a function of two variables for which the first- and second-order partial derivatives are continuous around (x_0,y_0). Suppose and . Define

(i) If and , then f has a

(ii) If and , then f has a

(iii) If , then f has a

(iv) If , then

Problem solving strategy

Problem:

Let $z = f(x, y)$ be a function of two variables for which the first- and second-ordered partial derivatives are continuous. Find local extrema.

Solution:

1. Determine critical points (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
   Discard any points where f_x or f_y does not exist.
2. Caclulate D for each critical point
3. Apply the Second derivative test to determine if (x_0, y_0) is a local minimum, local maximum or a saddle point.

Local extrema. Examples

Find the critical points for the following function and use the second derivative test to find the local extrema

$$f(x,y) = x^3 + 2xy - 2x - 4y$$

Step 1: Compute ∇f and find the critical points

$f_x =$

$f_y =$

f_x and f_y are

Find (x,y) such that

$$\begin{cases} \\ \\ \end{cases}$$

Function f has

Local extrema. Examples

Step 2: Compute

Start by computing f_{xx}, f_{xy}, f_{yx}, f_{yy} at $(2,-5)$

$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}$, $f_{xx}(2,-5) =$

$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}$, $f_{xy}(2,-5) =$

$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}$, $f_{yy}(2,-5) =$

$D =$

Step 3: Second derivative test.

Local extrema. Examples

Find the critical points for the following function and use the second derivative test to find the local extrema

$$f(x,y) = xy\, e^{-\frac{x^2+y^2}{2}}$$

Step 1

$f_x =$

$f_y =$

f_x and f_y are defined for all (x,y)

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Leftrightarrow$$

Critical points:

Local extrema. Examples

Step 2 Second order partial derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left[y(1-x^2)e^{-\frac{x^2+y^2}{2}} \right] =$$

$$f_{yy} =$$

$$f_{xy} =$$

Local extrema. Examples

$$f_{xx} = -xy(3-x^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{yy} = -xy(3-y^2)e^{-\frac{x^2+y^2}{2}}$$

Consider the critical point $(1,1)$

$f_{xx}(1,1) =$

$f_{xy}(1,1) =$

$f_{yy}(1,1) =$

$D =$

Local extrema. Examples

$$f_{xx} = -xy(3-x^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{yy} = -xy(3-y^2)e^{-\frac{x^2+y^2}{2}}$$

Consider the critical point $(1,-1)$

$f_{xx}(1,1) =$

$f_{xy}(1,1) =$

$f_{yy}(1,1) =$

$D =$

Local extrema. Examples

$$f_{xx} = -xy(3-x^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_{yy} = -xy(3-y^2)e^{-\frac{x^2+y^2}{2}}$$

Consider the critical point $(0,0)$

$f_{xx}(0,0) =$

$f_{xy}(0,0) =$

$f_{yy}(0,0) =$

$D =$