MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Vectors in the plane.
Vectors in three dimensions
Next: Strang 2.3

Week 1:

- check the course website
- homework 1 (due Friday, September 30) $\vec{v} = \langle x, y \rangle$
- join Piazza, Edfinity

Last time

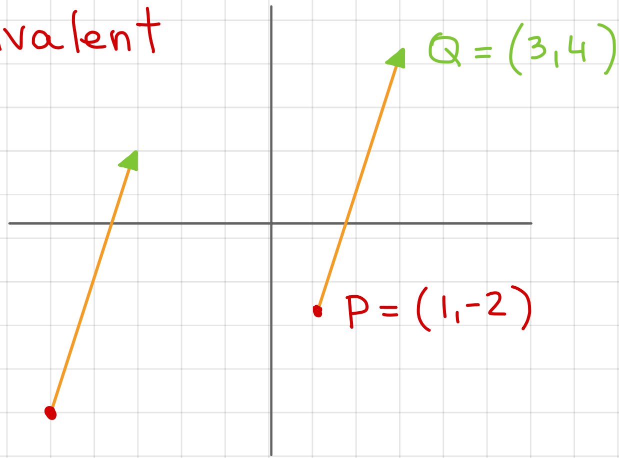
Def. A vector is a quantity that has both
magnitude (size, length) and direction

Forces, displacements, velocity are described by vectors.

A vector in a plane is represented by a directed line segment from the initial point to the terminal point.

We say that $\vec{v}$ and $\vec{w}$ are equivalent if they have the same direction and magnitude (denoted $\vec{v} = \vec{w}$).

We treat equivalent vectors as equal.



Scalar multiplication

magnitude
no direction

Let $\vec{v}$ be a vector and k be a real number (scalar)

Then $k\vec{v}$, called the scalar product of k and $\vec{v}$,

is a vector such that

$$\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$$

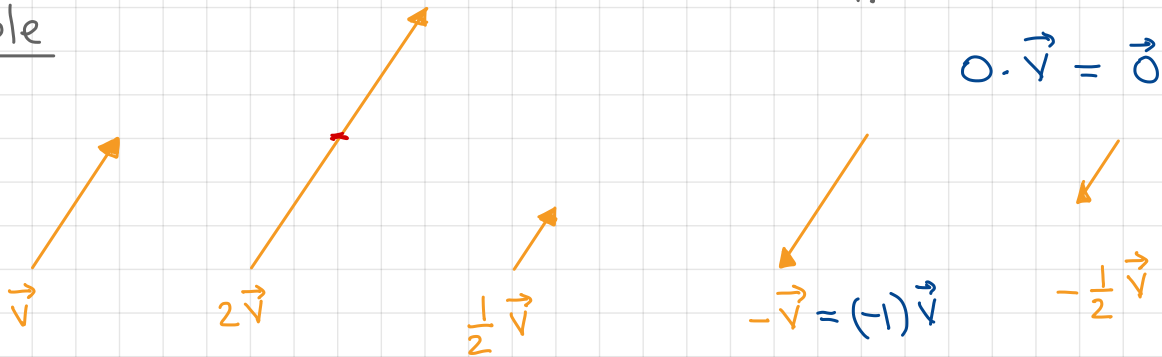
$k\vec{v}$ has the same direction as $\vec{v}$ if $k>0$

$k\vec{v}$ has the direction opposite to the direction of $\vec{v}$ if $k<0$

If $k=0$ or $\vec{v}=0$, then

$$0 \cdot \vec{v} = \vec{0}$$

Example

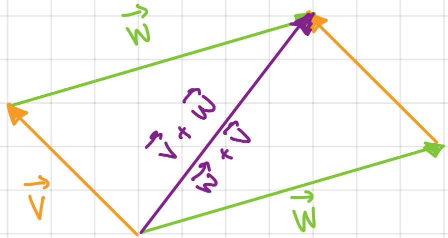


Vector addition

Let $\vec{v}$ and $\vec{w}$ be two vectors. Place the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. Then the vector with initial point at the initial point of $\vec{v}$ and the terminal point at the terminal point of $\vec{w}$ is called the vector sum of $\vec{v}$ and $\vec{w}$, and is denoted $\vec{v} + \vec{w}$.

Example

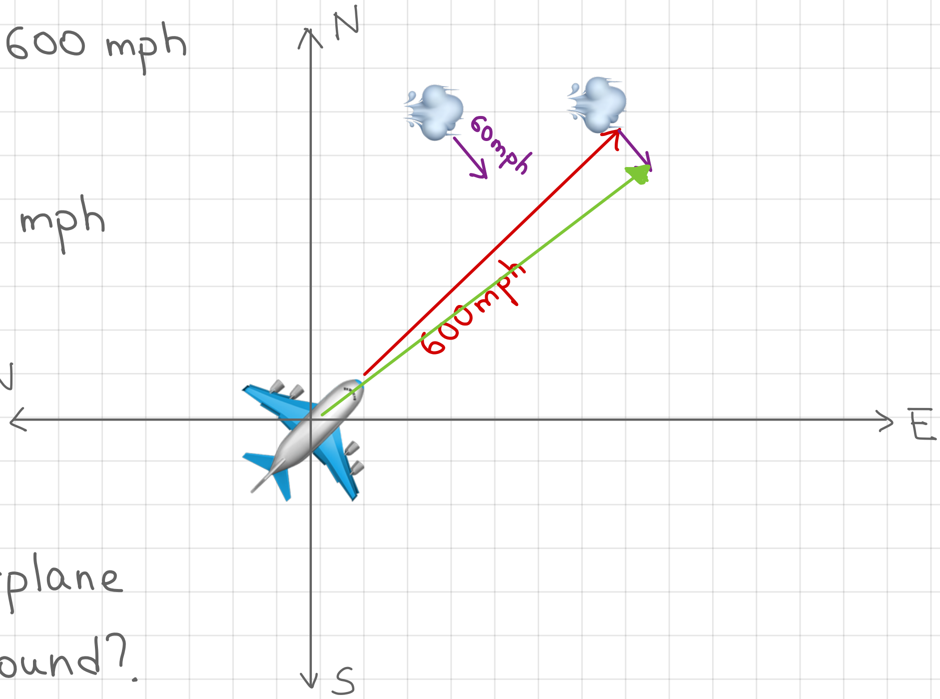
Notice that $\vec{v} + \vec{w} = \vec{w} + \vec{v}$



Definition of a vector

Airplane flies NE at 600 mph
(relative to the air)

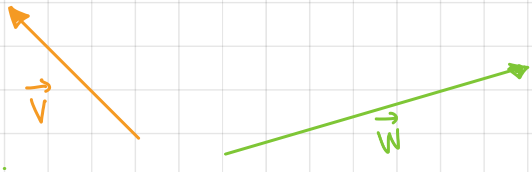
Wind blows SE at 60 mph



How fast does the airplane fly relative to the ground?
In what direction?

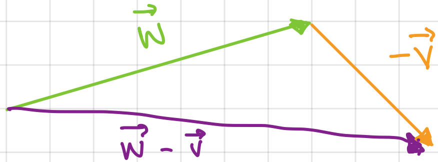
Combining vectors

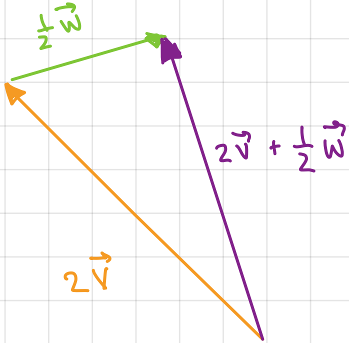
We know how to define (geometrically) $k_1\vec{v}_1 + k_2\vec{v}_2$ or $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 \ \dots$ (linear combination of vectors)



Example

- $\vec{w} - \vec{v} := \vec{w} + (-1)\vec{v}$

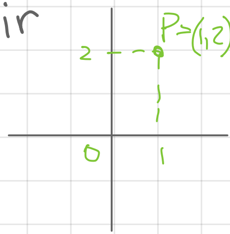


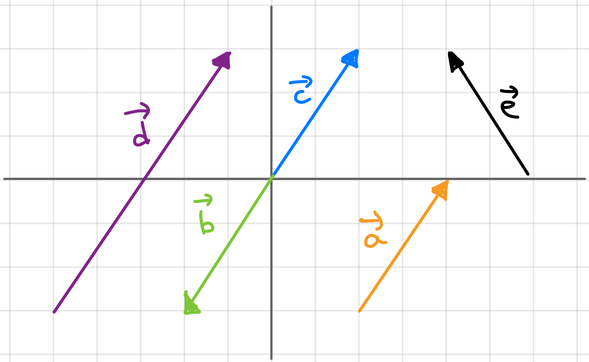


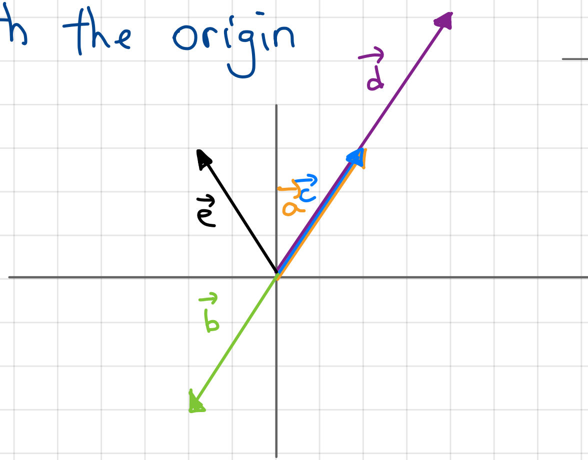
- $2\vec{v} + \frac{1}{2}\vec{w}$

Vector components

It is easier to work with vectors in a coordinate system. Since the location of the initial vector does not matter, let's place all vectors in the plane so that their initial points coincide with the origin



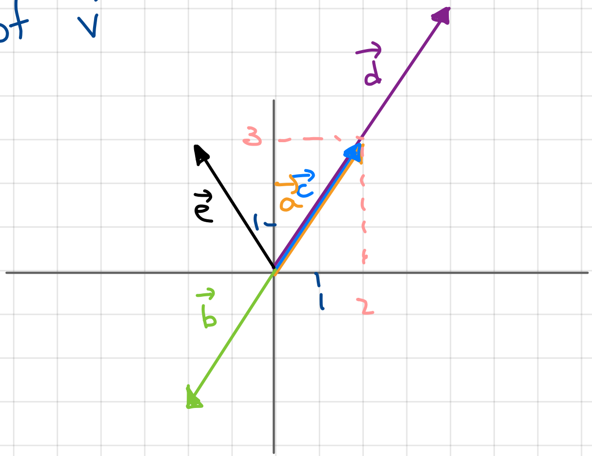




We call such vectors standard-positioned vectors, and they can be described by the coordinates of the terminal points

Vector components

<u>Def.</u> The vector with initial point $(0,0)$ and the terminal point (x,y) can be written in component form as $\vec{v} = \langle x, y \rangle$. The scalars x and y are called the components of $\vec{v}$



$\vec{a} = \langle 2, 3 \rangle$

$\vec{b} = \langle -2, -3 \rangle$

$\vec{d} = \langle 4, 6 \rangle$

$\vec{e} = \langle -2, 3 \rangle$

Vector components

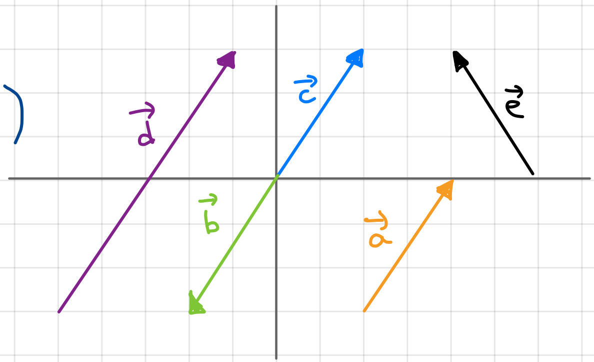
If the vector is not in standard position, but we know the coordinates of its initial and terminal points, then we can find the vectors coordinates using the following rule:

Let $P = (x_i, y_i)$ and $Q = (x_t, y_t)$. Then

$$\overrightarrow{PQ} = \langle x_t - x_i , y_t - y_i \rangle$$

$\vec{a}$: $P = (2, -3)$, $Q = (4, 0)$

$$\vec{a} = \overrightarrow{PQ} = \langle 4-2, 0-(-3) \rangle$$
$$= \langle 2, 3 \rangle$$



Magnitude of the vector

Magnitude of the vector is the distance between its initial and terminal points.

If $P = (x_i, y_i)$, $Q = (x_t, y_t)$, then $\|\overrightarrow{PQ}\| = \sqrt{(x_t - x_i)^2 + (y_t - y_i)^2}$

If $\vec{v} = \langle x, y \rangle$, then $\|\vec{v}\| = \sqrt{x^2 + y^2}$

Example

- $P = (2, -3)$, $Q = (4, 0)$

  $\|\overrightarrow{PQ}\| = \sqrt{(4-2)^2 + (0-(-3))^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$

- $\vec{a} = \langle 2, 3 \rangle$

  $\|\vec{a}\| = \sqrt{2^2 + 3^2} = \sqrt{13}$

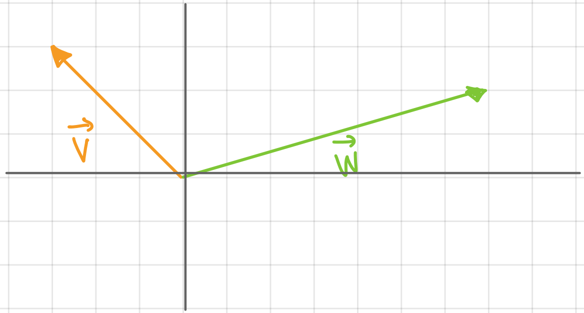
Vector operations in component form

<u>Def</u>. Let $\vec{v} = \langle x_1, y_1 \rangle$, $\vec{w} = \langle x_2, y_2 \rangle$, $k \in \mathbb{R}$.

Then

- $k\vec{v} = \langle kx_1, ky_1 \rangle$ (scalar multiplication)
- $\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2 \rangle$ (vector addition)

<u>Example</u>



$\vec{v} = \langle -3, 3 \rangle$

$\vec{w} = \langle 7, 2 \rangle$

- $\vec{v} + \vec{w} = \langle -3+7, 3+2 \rangle = \langle 4, 5 \rangle$
- $\vec{w} - \vec{v} = \langle 7-(-3), 2-3 \rangle = \langle 10, -1 \rangle$
- $2\vec{v} + \frac{1}{2}\vec{w} = \langle 2\cdot(-3) + \frac{1}{2}\cdot 7, 2\cdot 3 + \frac{1}{2}\cdot 2 \rangle \simeq \langle -2.5, 7 \rangle$

Properties of vector operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in the plane. Let r, s be scalars. Then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative property)

(iii) $\vec{u} + \vec{0} = \vec{u}$ (additive identity property)

(iv) $\vec{u} + (-\vec{u}) = \vec{0}$ (additive inverse property)

(v) $r(s\vec{u}) = (rs)\vec{u}$ (associativity of scalar mult.)

(vi) $(r+s)\vec{u} = r\vec{u} + s\vec{u}$ (distributive property)

(vii) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$ (distributive property)

(viii) $1 \cdot \vec{u} = \vec{u}$, $0 \cdot \vec{u} = \vec{0}$ (identify and zero properties)