MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: The dot product

Next: Strang 2.4

Week 1:

- office hours schedule
- homework 1 (due Monday, October 3)
- join Piazza, Edfinity

Equation of a sphere

Example Find the standard equation of the sphere with center (2, 3, 4) and point (0, 11, -1)

In order to write the equation of a sphere we need to know the center (given) and the radius (unknown). Radius is the distance from the center of the sphere to any point of the sphere (in particular to (0, 11, -1))

Therefore,

$$r = \sqrt{(0-2)^2 + (11-3)^2 + (-1-4)^2}$$

$$= \sqrt{4 + 64 + 25}$$

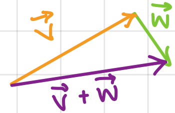
$$= \sqrt{93}$$

Equation of the sphere : $(x-2)^2 + (y-3)^2 + (z-4)^2 = 93$

Vectors in $\mathbb{R}^3$

Complete analogy with vectors in the plane

- vectors are quantities with both magnitude and direction
- vectors are represented by directed line segments (arrows)
- vector is in the standard position if its initial point is $(0,0,0)$
- vectors admit the component representation $\vec{v} = \langle x, y, z \rangle$
- $\vec{0} = \langle 0, 0, 0 \rangle$
- vector addition and scalar multiplication are defined analogously to plane vectors :



- in the component form :

$$k_1 \langle x_1, y_1, z_1 \rangle + k_2 \langle x_2, y_2, z_2 \rangle = \langle k_1 x_1 + k_2 x_2 , k_1 y_1 + k_2 y_2 , k_1 z_1 + k_2 z_2 \rangle$$

- $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$ are standard unit vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$

- if $\vec{v} = \langle x, y, z \rangle$, then $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ (standard unit form)
- if $P = (x_i, y_i, z_i)$, $Q = (x_t, y_t, z_t)$, then $\overrightarrow{PQ} = \langle x_t - x_i, y_t - y_i, z_t - z_i \rangle$
- if $\vec{v} = \langle x, y, z \rangle$, then $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$
- to find the unit vector in the direction $\vec{v} = \langle x, y, z \rangle$,

  multiply $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$: $\vec{u} = \left\langle \frac{x}{\|\vec{v}\|}, \frac{y}{\|\vec{v}\|}, \frac{z}{\|\vec{v}\|} \right\rangle$

Example Let $P = (0, 3, -2)$, $Q = (2, 2, 2)$. Express $\overrightarrow{PQ}$ in component form and in standard unit form.

$$\overrightarrow{PQ} = \langle 2-0, 2-3, 2-(-2) \rangle = \langle 2, -1, 4 \rangle = 2\vec{i} - \vec{j} + 4\vec{k}$$

Example Let $\vec{v} = \langle 2, 0, 6 \rangle$, $\vec{w} = \langle 1, -1, -2 \rangle$. Then

$$\vec{v} + 3\vec{w} = \langle 1\cdot 2 + 3\cdot 1, 1\cdot 0 + 3\cdot(-1), 1\cdot 6 + 3\cdot(-2) \rangle = \langle 5, -3, 0 \rangle$$

$$\|\vec{v} + 3\vec{w}\| = \sqrt{5^2 + (-3)^2 + 0^2} = \sqrt{25 + 9} = \sqrt{34}$$

Properties of vector operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Let r, s be scalars.

Then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative property)

(iii) $\vec{u} + \vec{0} = \vec{u}$ (additive identity property)

(iv) $\vec{u} + (-\vec{u}) = \vec{0}$ (additive inverse property)

(v) $r(s\vec{u}) = (rs)\vec{u}$ (associativity of scalar mult.)

(vi) $(r+s)\vec{u} = r\vec{u} + s\vec{u}$ (distributive property)

(vii) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$ (distributive property)

(viii) $1 \cdot \vec{u} = \vec{u}$, $0 \cdot \vec{u} = \vec{0}$ (identify and zero properties)

Dot product (scalar product) of vectors

Def If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ are two vectors in $\mathbb{R}^3$, then the dot product or the scalar product of $\vec{v}$ and $\vec{w}$ is given by the sum of products of vector components

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

(in $\mathbb{R}^2$ $\vec{v} = \langle v_1, v_2 \rangle$
$\vec{u} = \langle u_1, u_2 \rangle$
$\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2$

Examples

$\vec{v} = \langle 0, 1, -2 \rangle$, $\vec{w} = \langle 5, 6, 7 \rangle$, $\vec{v} \cdot \vec{w} = 0 \cdot 5 + 1 \cdot 6 + (-2) \cdot 7 = -8$

$\vec{p} = \vec{j} - \vec{k}$, $\vec{q} = \vec{i} + 2\vec{j} + 2\vec{k}$, $\vec{p} \cdot \vec{q} = 0 \cdot 1 + 1 \cdot 2 + (-1) \cdot 2 = 0$

Dot (scalar) product takes two vectors and returns a number

Dot product

Theorem 2.3 (Properties of the dot product)

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors and let c be a scalar.

Then (i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative)

(ii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive)

(iii) $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ (associative)

(iv) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ (magnitude)

Proof. (iv) Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

$$\vec{v} \cdot \vec{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$$= v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3 = v_1^2 + v_2^2 + v_3^2 = \|\vec{v}\|^2$$

(i) $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}$

Dot product

Example

$\vec{a} = \langle -2, 2, 1 \rangle$, $\vec{b} = \langle -2, -5, 1 \rangle$, $\vec{c} = \langle 0, 3, -1 \rangle$

$$\vec{a}\,(\vec{b}\cdot\vec{c}) = \langle -2, 2, 1 \rangle \big((-2)\cdot 0 + (-5)\cdot 3 + 1\cdot(-1) \big)$$
$$= \langle -2, 2, 1 \rangle \cdot (-16) = \langle 32, -32, -16 \rangle$$

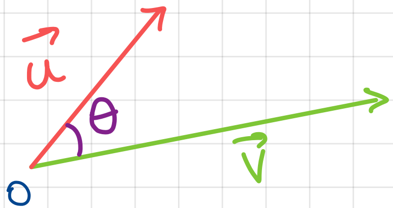
$$\vec{a}\cdot(2\vec{c}) = \langle -2, 2, 1 \rangle \cdot \langle 0, 6, -2 \rangle = (-2)\cdot 0 + 2\cdot 6 + 1\cdot(-2)$$
$$= 10$$

Angle between two vectors

Dot product provides a convenient way to measure the angle between two vectors.

Theorem 2.4

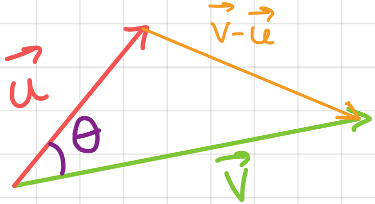
If



, $0 \le \theta \le \pi$,

then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

Proof Consider vector $\vec{v} - \vec{u}$.



Law of cosines:

$$\vec{v}\cdot(\vec{v}-\vec{u}) - \vec{u}\cdot(\vec{v}-\vec{u}) = (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u}) = \|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$\|$$

$$\vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{u} = \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v} + \|\vec{u}\|^2 \quad \Big/ \Rightarrow \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

Dot product and angles between vectors

From Theorem 2.4 we have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}, \quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}\right)$$