MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: The dot product

Next: Strang 2.4

Week 1:

- office hours schedule
- homework 1 (due Monday, October 3)
- join Piazza, Edfinity

Equation of a sphere

Example Find the standard equation of the sphere with center $(2,3,4)$ and point $(0,11,-1)$

In order to write the equation of a sphere we need to know the center (given) and the radius (unknown). Radius is the distance from the center of the sphere to any point of the sphere (in particular to $(0,11,-1)$)

Therefore,

$$r =$$

$$=$$

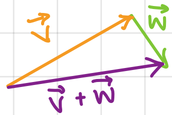
$$=$$

Equation of the sphere :

Vectors in $\mathbb{R}^3$

Complete analogy with vectors in the plane

- vectors are quantities with both magnitude and direction
- vectors are represented by directed line segments (arrows)
- vector is in the standard position if its initial point is $(0,0,0)$
- vectors admit the component representation $\vec{v} = \langle x, y, z \rangle$
- $\vec{0} = \langle 0, 0, 0 \rangle$
- vector addition and scalar multiplication are defined analogously to plane vectors:



- in the component form:

$$k_1 \langle x_1, y_1, z_1 \rangle + k_2 \langle x_2, y_2, z_2 \rangle = \langle k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2, k_1 z_1 + k_2 z_2 \rangle$$

- $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$ are standard unit vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$

- if $\vec{v} = \langle x, y, z \rangle$, then $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ (standard unit form)
- if $P = (x_i, y_i, z_i)$, $Q = (x_t, y_t, z_t)$, then $\overrightarrow{PQ} = \langle x_t - x_i, y_t - y_i, z_t - z_i \rangle$
- if $\vec{v} = \langle x, y, z \rangle$, then $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$
- to find the unit vector in the direction $\vec{v} = \langle x, y, z \rangle$, multiply $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$: $\vec{u} = \left\langle \frac{x}{\|\vec{v}\|}, \frac{y}{\|\vec{v}\|}, \frac{z}{\|\vec{v}\|} \right\rangle$

Example Let $P = (0, 3, -2)$, $Q = (2, 2, 2)$. Express $\overrightarrow{PQ}$ in component form and in standard unit form.

$\overrightarrow{PQ} =$

Example Let $\vec{v} = \langle 2, 0, 6 \rangle$, $\vec{w} = \langle 1, -1, -2 \rangle$. Then

$\vec{v} + 3\vec{w} =$

$\|\vec{v} + 3\vec{w}\| =$

Properties of vector operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Let r, s be scalars.

Then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative property)

(iii) $\vec{u} + \vec{0} = \vec{u}$ (additive identity property)

(iv) $\vec{u} + (-\vec{u}) = \vec{0}$ (additive inverse property)

(v) $r(s\vec{u}) = (rs)\vec{u}$ (associativity of scalar mult.)

(vi) $(r+s)\vec{u} = r\vec{u} + s\vec{u}$ (distributive property)

(vii) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$ (distributive property)

(viii) $1 \cdot \vec{u} = \vec{u}$, $0 \cdot \vec{u} = \vec{0}$ (identity and zero properties)

Dot product (scalar product) of vectors

Def If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ are two vectors in $\mathbb{R}^3$, then the dot product or the scalar product of $\vec{v}$ and $\vec{w}$ is given by the sum of products of vector components

$$\vec{v} \bullet \vec{w} =$$

(in $\mathbb{R}^2$ $\vec{v} = \langle v_1, v_2 \rangle$
$\vec{u} = \langle u_1, u_2 \rangle$
$\vec{v} \cdot \vec{u} =$

Examples

$\vec{v} = \langle 0, 1, -2 \rangle$, $\vec{w} = \langle 5, 6, 7 \rangle$, $\vec{v} \cdot \vec{w} =$

$\vec{p} = \vec{j} - \vec{k}$, $\vec{q} = \vec{i} + 2\vec{j} + 2\vec{k}$, $\vec{p} \cdot \vec{q} =$

Dot (scalar) product takes two vectors and returns a number

Dot product

Theorem 2.3 (Properties of the dot product)

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors and let c be a scalar.

Then (i) (commutative)

(ii) (distributive)

(iii) (associative)

(iv) (magnitude)

Proof.

Dot product

Example

$\vec{a} = \langle -2, 2, 1 \rangle$, $\vec{b} = \langle -2, -5, 1 \rangle$, $\vec{c} = \langle 0, 3, -1 \rangle$

$\vec{a}\,(\vec{b} \cdot \vec{c}) =$

$=$

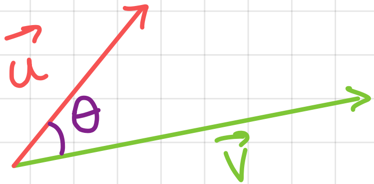
$\vec{a} \cdot (2\vec{c}) =$

Angle between two vectors

Dot product provides a convenient way to measure the angle between two vectors.

Theorem 2.4

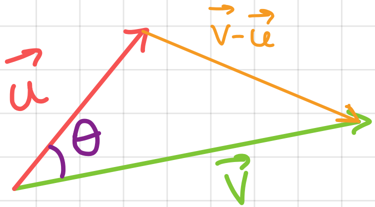
If



, $0 \le \theta \le \pi$,

then $\vec{u} \cdot \vec{v} =$

Proof Consider vector $\vec{v} - \vec{u}$.



Law of cosines:

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Dot product and angles between vectors

From Theorem 2.4 we have

$$\cos\theta = \qquad , \quad \theta =$$

Examples

Find the angle between $\vec{u}$ and $\vec{v}$

(a) $\vec{u} = -\vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = \vec{i} + 2\vec{j}$

$\|\vec{u}\| =$, $\|\vec{v}\| =$,

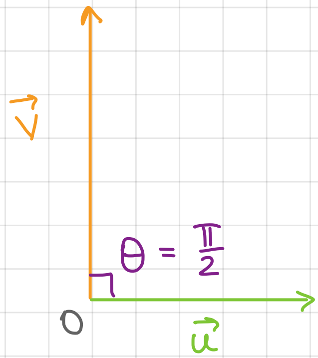
$\vec{u} \cdot \vec{v} =$ $\Rightarrow$ $\cos\theta =$, $\theta =$

(b) $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle -7, 2, 1 \rangle$

$\vec{u} \cdot \vec{v} =$ $\Rightarrow$ $\cos\theta =$ $\Rightarrow$ $\theta =$

Orthogonal vectors

If $\cos\theta = 0$, then $\theta = \frac{\pi}{2}$, which means that the vectors form a right angle



We call such vectors

Theorem 2.5

The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal

Example Determine whether $\vec{p} = \langle 1, 3, 0 \rangle$ and $\vec{q} = \langle -6, 2, 5 \rangle$ are orthogonal. Since $\vec{p} \cdot \vec{q} =$

we conclude that $\vec{p}$ and $\vec{q}$ are

Orthogonality of standard unit vectors

Recall: $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

Then $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$

$\vec{i} \cdot \vec{j} =$

$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$

We say that $\vec{i}, \vec{j}, \vec{k}$ are

<u>Example</u> $(10\vec{i} - \vec{j}) \cdot (-\vec{i} + 2\vec{k})$

$=$

$=$

$\langle 10,-1,0 \rangle \cdot \langle -1,0,2 \rangle =$

Using vectors to represent data

Fruit vendor sells apples, bananas and oranges. On a given day he sells 30 apples, 12 bananas and 18 oranges. Define the vector

$\vec{q} =$ (quantities)

Suppose that the vendor sets the following prices
0.5 per apple, 0.25 per banana, 1 per orange

Define the vector of prices

$\vec{p} =$

Then $\vec{q} \cdot \vec{p} =$

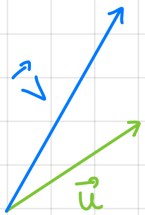
is vendor's

Projections

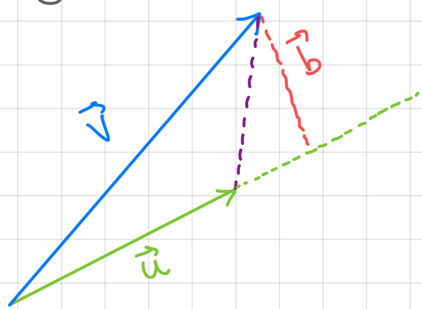
Let $\vec{u}$ and $\vec{v}$ be two vectors. Sometimes we want to decompose $\vec{v}$ into two components

$\vec{v} = \vec{a} + \vec{b}$ such that $\vec{a}$ is parallel to $\vec{u}$

and $\vec{b}$ is orthogonal to $\vec{u}$



Why?

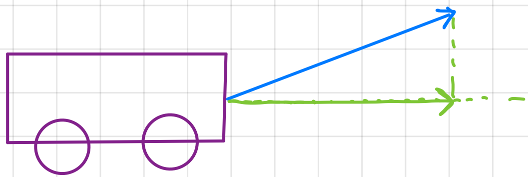


① Find the area of

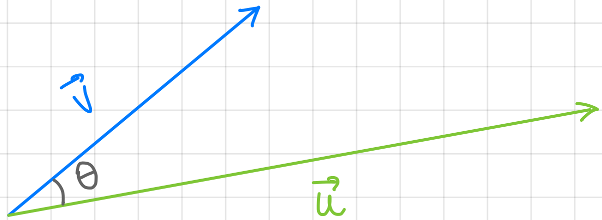
Area of this triangle is

② Child pulls a wagon

How much force is actually moving the wagon forward?



Projections



$\|\vec{a}\| =$

$\vec{a} =$

$=$

<u>Def</u> (Projection). The vector projection of $\vec{v}$ onto $\vec{u}$ is the vector labeled $\text{proj}_{\vec{u}} \vec{v}$ given by

$$\text{proj}_{\vec{u}} \vec{v} =$$

The length of $\text{proj}_{\vec{u}} \vec{v}$, $\|\text{proj}_{\vec{u}} \vec{v}\| = \dfrac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\|} =: \text{comp}_{\vec{u}} \vec{v}$

is called the scalar projection of $\vec{v}$ onto $\vec{u}$

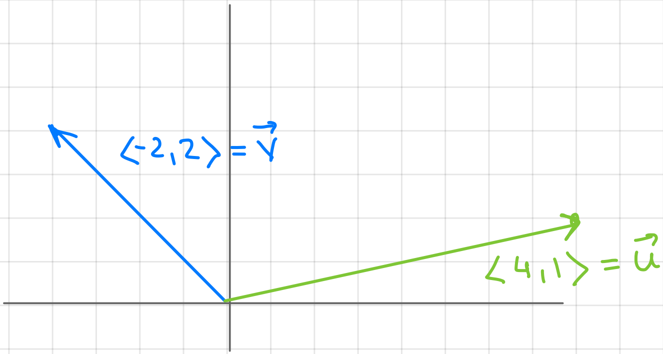
Projections

Let $\vec{v}$ and $\vec{u}$ be nonzero vectors. Then

$\vec{u}$ and $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$ are

$$\vec{u} \cdot \left(\vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \right) =$$

Example Find the projection of $\langle -2, 2 \rangle$ onto $\langle 4, 1 \rangle$



$$\text{proj}_{\vec{u}}\vec{v} =$$

$$=$$

$$=$$

Projections

Example Find the projection of $\langle -2, 2, 3 \rangle$ onto $\langle 10, -1, 0 \rangle$

$$\text{proj}_{\vec{u}}\, \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\, \vec{u}$$

$$=$$

$$=$$