MATH 10C: Calculus III (Lecture B00)

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Today: The cross product

Next: Strang 2.4

Week 2:

- homework 2 (due Monday, October 10)
- survey on Canvas Quizzes (due Friday, October 7)

The cross product

Def Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Then the cross product of $\vec{u}$ and $\vec{v}$ is vector

$$\vec{u} \times \vec{v} =$$

$$=$$

Example

$$\vec{p} = \langle 1, 2, 3 \rangle, \quad \vec{q} = \langle -1, 2, 0 \rangle$$

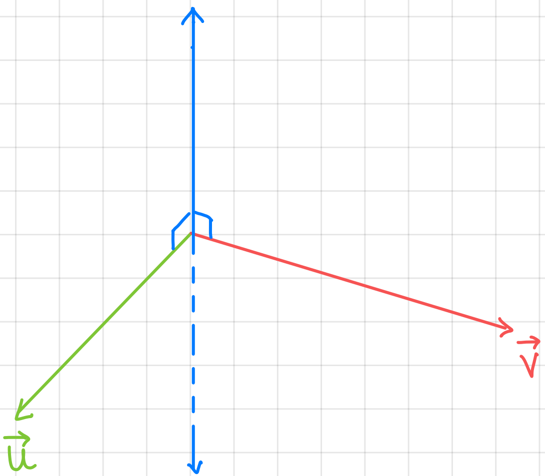
$$\vec{p} \times \vec{q} =$$

$$=$$

$$\vec{p} \cdot (\vec{p} \times \vec{q}) = \qquad , \quad \vec{q} \cdot (\vec{p} \times \vec{q}) =$$

The cross product

Fact: Vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$!



and the direction is determined by the right-hand rule.

Indeed,

$\vec{p} = \langle 1, 2, 3 \rangle$, $\vec{q} = \langle -1, 2, 0 \rangle$, $\vec{p} \times \vec{q} = \langle -6, -3, 4 \rangle$

$\vec{q} \times \vec{p} =$

$=$

Properties of the cross product

Exercise $\vec{i} \times \vec{j} = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle =$

$=$

$\vec{i} \times \vec{j} =$ $\qquad$ $\vec{j} \times \vec{k} =$ $\qquad$ $\vec{k} \times \vec{i} =$

Theorem 2.6 Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Then

(i) $\vec{u} \times \vec{v} =$

(ii) $\vec{u} \times (\vec{v} + \vec{w}) =$

(iii) $c(\vec{u} \times \vec{v}) =$

(iv) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} =$

(v) $\vec{v} \times \vec{v} =$

(vi) $\vec{u} \cdot (\vec{v} \times \vec{w}) =$

For proof expand both sides in terms of components of $\vec{u}, \vec{v}, \vec{w}$

Properties of cross product

In general, $(\vec{u}\times\vec{v})\times\vec{w} \neq \vec{u}\times(\vec{v}\times\vec{w})$

$$(\vec{i}\times\vec{i})\times\vec{j} =$$

$$\vec{i}\times(\vec{i}\times\vec{j}) =$$

Example (a) Calculate $(2\vec{i})\cdot\left((3\vec{j})\times\vec{k} + \vec{i}\times(-4)\vec{k}\right)$

$$(2\vec{i})\times\left((3\vec{j})\times\vec{k} + \vec{i}\times(-4)\vec{k}\right) =$$

$$=$$

$$=$$

(b) Show that $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot(\vec{u}\times\vec{v}) \overset{(vi)}{=}$$

$$\vec{v}\cdot(\vec{u}\times\vec{v}) =$$

Magnitude of the cross product

Fact. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

Proof. Expand both sides using components $\vec{u} = \langle u_1, u_2, u_3 \rangle$
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

Theorem 2.7 Let $\vec{u}$ and $\vec{v}$ be vectors, let θ be the angle between them. Then

$$\|\vec{u} \times \vec{v}\| =$$

Proof From (*) $\|\vec{u} \times \vec{v}\|^2 =$

From Thm. 2.4 $\vec{u} \cdot \vec{v} =$ Then

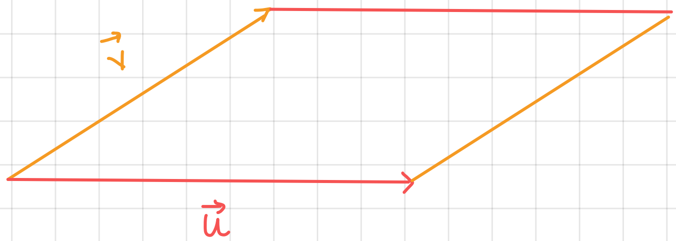
$\|\vec{u}\| \cdot \|\vec{v}\| =$

Geometric interpretation

Summary: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$.
Then $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ such that

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$ (right-hand rule)
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$ with θ = angle between $\vec{u}$ and $\vec{v}$

Consider a parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$

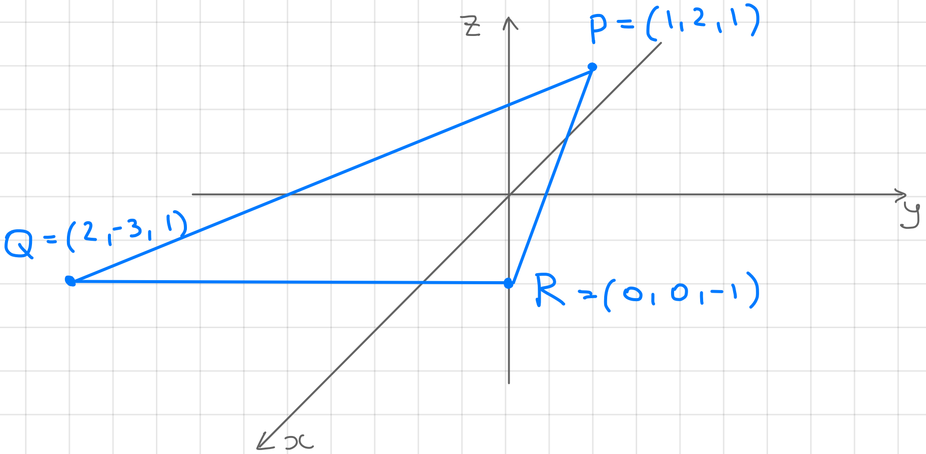


Area (▱) =

Conclusion: magnitude of $\vec{u} \times \vec{v}$ is equal to the

Example

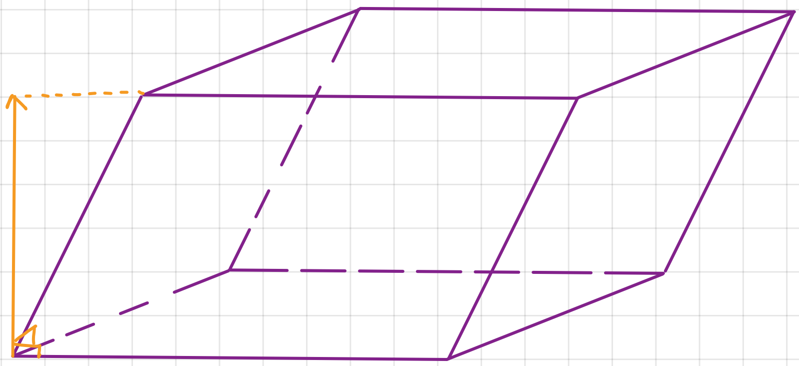
Let $P=(1,2,1)$, $Q=(2,-3,1)$, $R=(0,0,-1)$ be the vertices on a triangle. Find its area.



$\vec{PQ} =$, $\vec{PR} =$, $\text{Area}(\Delta) =$

$\vec{PQ} \times \vec{PR} =$, $\|\vec{PQ} \times \vec{PR}\| =$

Volume of a parallelepiped



Three-dimensional prism with six facets that are each parallelograms.

Volume =

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$, consider a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

Area of the base =

Height =

Volume of a parallelepiped

<u>Definition</u> The triple scalar product of $\vec{u}, \vec{v}$ and $\vec{w}$ is given by

<u>Theorem 2.10</u> The volume of a parallelepiped given by vectors $\vec{u}, \vec{v}, \vec{w}$ is the absolute value of the triple scalar product

<u>Example</u> Find the volume of the parallelepiped with adjacent edges (spanned by) $\vec{u} = \langle -1, -2, 1 \rangle$, $\vec{v} = \langle 4, 3, 2 \rangle$, $\vec{w} = \langle 0, -5, -2 \rangle$

$\vec{v} \times \vec{w} =$, $\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle -1, -2, 1 \rangle \cdot \langle 4, 8, -20 \rangle =$

$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| =$

Summary

Dot (scalar) product: $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

- characterizes the angle $0 \le \theta \le \pi$ between $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

Cross (vector) product: $\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$

- gives a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$
- its length gives the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ $\quad \|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\cdot\sin\theta$

Triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$: $\vec{u}\cdot(\vec{v}\times\vec{w})$

- its absolute value gives the volume of the parallelepiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Last remark

If you know how to compute the determinant of a 3×3 matrix, then the cross product of $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ can be computed as

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\,(u_2 v_3 - u_3 v_2) - \vec{j}\,(u_1 v_3 - u_3 v_1) + \vec{k}\,(u_1 v_2 - u_2 v_1)$$

Similarly, the triple scalar product of $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ can be computed as

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 (v_2 w_3 - v_3 w_2) - u_2 (v_1 w_3 - v_3 w_1) + u_3 (v_1 w_2 - v_2 w_1)$$