MATH 10C: Calculus III (Lecture B00)

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Today: Equations of lines and planes

Next: Strang 3.1

Week 2:

- homework 1 (due Monday, October 3)
- survey on Canvas Quizzes (due Friday, October 7)

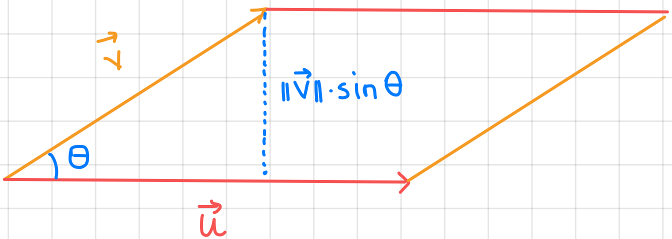
Cross product

Summary: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$.

Then $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ such that

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$ (right-hand rule)
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$ with θ = angle between $\vec{u}$ and $\vec{v}$

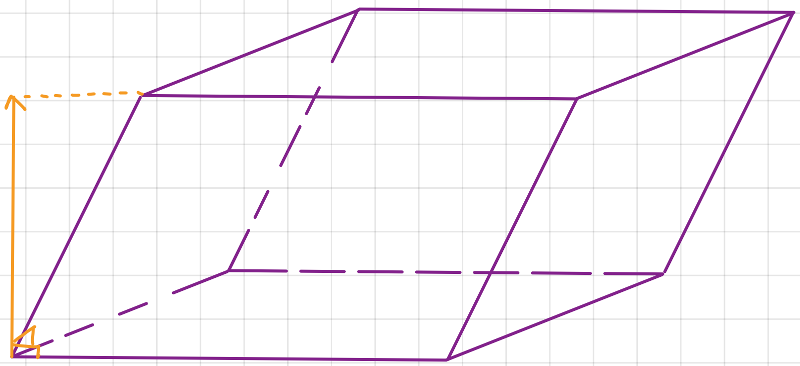
Consider a parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$



$$\text{Area}(\text{▱}) = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$$
$$= \|\vec{u} \cdot \vec{v}\|$$

Conclusion: magnitude of $\vec{u} \times \vec{v}$ is equal to the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$

Volume of a parallelepiped



Three-dimensional prism with six facets that are each parallelograms.

Volume =

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$, consider a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

Area of the base =

Height =

Volume of a parallelepiped

Definition The triple scalar product of $\vec{u}, \vec{v}$ and $\vec{w}$ is given by

Theorem 2.10 The volume of a parallelepiped given by vectors $\vec{u}, \vec{v}, \vec{w}$ is the absolute value of the triple scalar product

Example Find the volume of the parallelepiped with adjacent edges (spanned by) $\vec{u} = \langle -1, -2, 1\rangle$, $\vec{v} = \langle 4, 3, 2\rangle$, $\vec{w} = \langle 0, -5, -2\rangle$

$\vec{v} \times \vec{w} =$, $\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle -1, -2, 1\rangle \cdot \langle 4, 8, -20\rangle =$

$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| =$

Summary

Dot (scalar) product: $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

- characterizes the angle $0 \leq \theta \leq \pi$ between $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

Cross (vector) product: $\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$

- gives a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$
- its length give the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ $\quad \|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\cdot\sin\theta$

Triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$: $\vec{u}\cdot(\vec{v}\times\vec{w})$

- its absolute value gives the volume of the parallelepiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Last remark

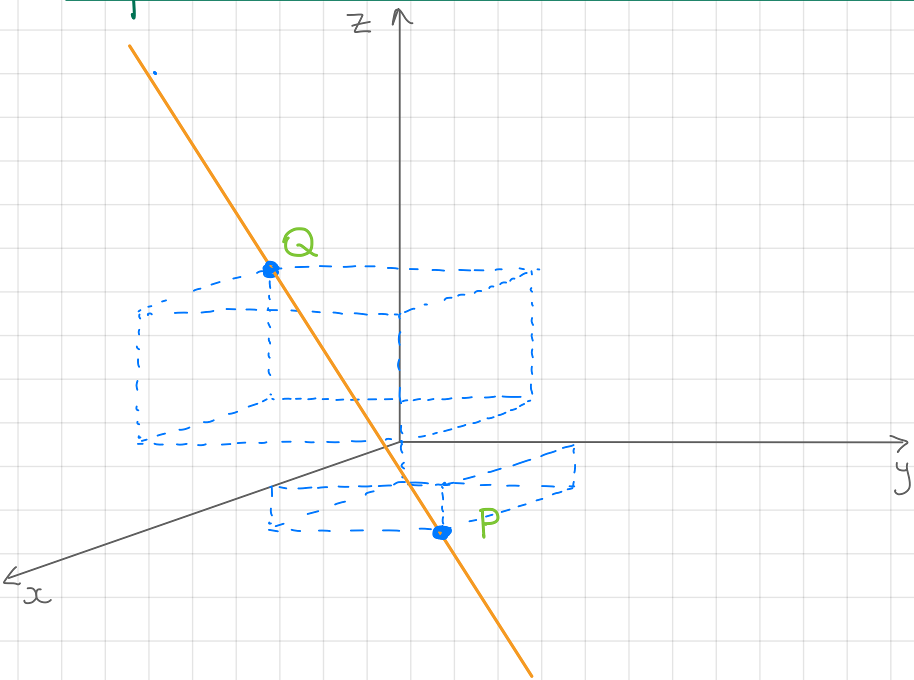
If you know how to compute the determinant of a 3×3 matrix, then the cross product of $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ can be computed as

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\,(u_2v_3-u_3v_2)-\vec{j}\,(u_1v_3-u_3v_1)+\vec{k}\,(u_1v_2-u_2v_1)$$

Similarly, the triple scalar product of $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$ can be computed as

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1(v_2w_3-v_3w_2)-u_2(v_1w_3-v_3w_1)+u_3(v_1w_2-v_2w_1)$$

Equation for a line in space



To describe a line in $\mathbb{R}^3$ we must know either (a) two points on the line, or (b) one point and direction.

Let L be a line passing through points P and Q.

Point R belongs to L if , i.e., either $\overrightarrow{PR}$ has the same direction as $\overrightarrow{PQ}$, or $\overrightarrow{PR}$ has direction opposite to $\overrightarrow{PQ}$ (or $\overrightarrow{PR} = \vec{0}$).

Equation for a line in space

Vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if

(by convention $\vec{0}$ is parallel to all vectors)

Given two distinct points P and Q, the line through P and Q is the collection of points R such that

Similarly, given point P and vector $\vec{v}$, the line through P with direction vector $\vec{v}$ is the collection of points R such that

(*)

Equation for a line in space

Let $P = (x_0, y_0, z_0)$, $R = (x, y, z)$ and $\vec{v} = \langle a, b, c \rangle$. Then

(*) implies

(**)

By equating components, we get that the coordinates of R (point on the line) satisfy the equations

(***)

If we denote $\vec{r} := \langle x, y, z \rangle$ and $\vec{r_0} := \langle x_0, y_0, z_0 \rangle$, then from (**)

(****)

Equation for a line in space

If a, b and c are all nonzero, we can rewrite (***)

which (since t can be any real number) is equivalent to

(*****)

Thm 2.11 (Parametric and symmetric eqs. of a line)

A line parallel to vector $\vec{v} = \langle a, b, c \rangle$ and passing through $P = (x_0, y_0, z_0)$ can be described by the following parametric equations:

If a, b and c are all nonzero, L can be described by the symmetric equation

Examples

Find parametric and symmetric equations of the line L passing through points $P = (3, 2, 1)$ and $Q = (5, 1, -2)$

First, identify the direction vector ($\overrightarrow{PQ}$ or $\overrightarrow{QP}$)

Take a point on the line (either P or Q).

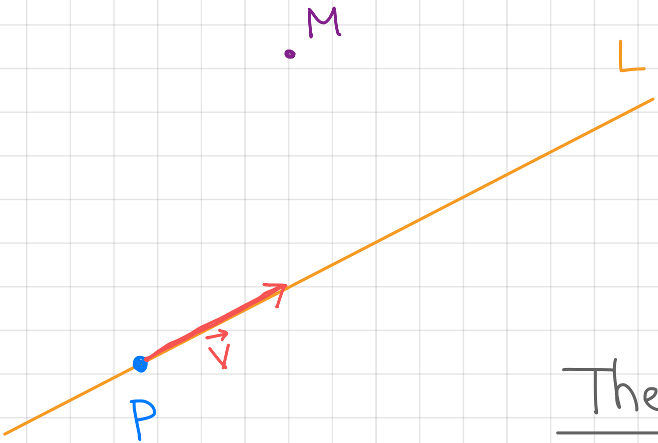
Parametric equation :

Symmetric equation :

Distance between a point and a line

Consider the line L through point P with direction vector $\vec{v}$.

Suppose M is not on the line. What is the distance between L and M?



Theorem 2.12. Let L be a line passing through P with direction vector $\vec{v}$. If M is any point not on L, then

Distance between a point and a line

Example

Find the distance between $M = (3, 2, 1)$ and the line $\frac{x-5}{2} = \frac{y+2}{2} = -z$

Identify a point on the line.

Identify the direction vector of the line:

Compute

Finally,

Relationships between lines in $\mathbb{R}^3$

Let L_1 and L_2 be two lines in $\mathbb{R}^3$. Then the following four possibilities exist:

		L_1 and L_2 share a common point	
		YES	NO
Direction vectors of L_1 and L_2 are parallel	YES	Equal	Parallel but not equal
	NO	Intersecting	Skew = not parallel not intersecting

Relationships between lines in $\mathbb{R}^3$

Example

L_1: direction vector $\vec{v}_1 = \langle 1, 2, 0 \rangle$, passing through $P_1 = (0,0,1)$

L_2: direction vector $\vec{v}_2 = \langle -3, -6, 0 \rangle$, passing through $P_2 = (1,2,3)$

L_3: direction vector $\vec{v}_3 = \langle 1, -1, 1 \rangle$, passing through $P_3 = (-1,4,-1)$

① L_1 and L_2 $\vec{v}_1$ parallel to $\vec{v}_2$, , therefore,

L_1 and L_2 are either

Write equations for L_1 :

Relationships between lines in $\mathbb{R}^3$

Example ② L_1 and L_3

(i) $\vec{v}_1 = \langle 1, 2, 0 \rangle$, $\vec{v}_3 = \langle 1, -1, 1 \rangle$. Are $\vec{v}_1$ and $\vec{v}_3$ parallel?

Parallel if and only if

$\Bigg\{$ this system has no solutions, so direction vectors are

L_1 and L_3 are

(ii) Do L_1 and L_3 have a point in common?

If $Q = (x, y, z)$ belongs to both L_1 and L_3, then the coordinates of Q must satisfy both equations

Relationships between lines in $\mathbb{R}^3$

Example

$$\begin{cases} x = t, \\ y = 2t, \\ z = 1, \end{cases} \quad \text{and} \quad \begin{cases} x = -1 + s \\ y = 4 - s \\ z = -1 + s \end{cases} \quad \text{for some } s, t \in \mathbb{R}$$

Equate the right-hand sides of the above equations

$$\Bigg\{ \qquad \qquad$$

If this system has a solution then L_1 and L_3 intersect

From the last equation we have . Substituting into the first two equations gives