MATH 10C (Lecture B00 - NEMISH) MIDTERM 1 10/19/22, 1-1:50pm, CENTR 115

SOLUTIONS

Name (last, first): ____________________

Student ID: ____________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

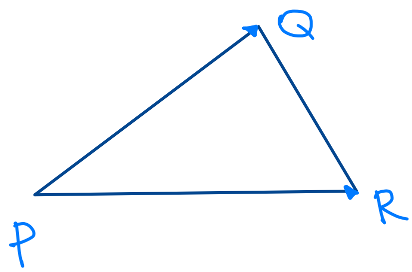
☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (20 points) Compute the area of the triangle with vertices

$$P = (1, 0, -1), \quad Q = (0, 1, 0), \quad R = (1, 0, 0).$$



The area of the triangle PQR is equal to $\frac{1}{2}$ of the area of the parallelogram spanned by the vectors $\vec{PQ}$ and $\vec{PR}$.

The area of the parallelogram is $\|\vec{PQ} \times \vec{PR}\|$.

Compute the components of $\vec{PQ}$ and $\vec{PR}$

$$\vec{PQ} = \langle -1, 1, 1 \rangle, \quad \vec{PR} = \langle 0, 0, 1 \rangle$$

Compute the cross product

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \vec{i} \cdot 1 - \vec{j}(-1) + \vec{k} \cdot 0 = \langle 1, 1, 0 \rangle$$

Compute the magnitude of $\vec{PQ} \times \vec{PR}$

$$\|\vec{PQ} \times \vec{PR}\| = \|\langle 1, 1, 0 \rangle\| = \sqrt{1+1} = \sqrt{2}$$

Area of the triangle PQR is $\frac{\sqrt{2}}{2}$

2. (20 points) Let

$$P = (1, 0, 2), \quad Q = (1, 1, 4), \quad R = (4, 2, \alpha)$$

be points in $\mathbb{R}^3$.

Determine the real number α such that vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are orthogonal.

Vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are orthogonal if $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 0$.

Compute the components of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$$\overrightarrow{PQ} = \langle 0, 1, 2 \rangle, \quad \overrightarrow{PR} = \langle 3, 2, \alpha - 2 \rangle$$

Compute the dot product with unknown α

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \langle 0, 1, 2 \rangle \cdot \langle 3, 2, \alpha - 2 \rangle$$

$$= 2 + 2(\alpha - 2) = 2\alpha - 2$$

Find α for which $\overrightarrow{PQ} \cdot \overrightarrow{PR}$ by solving

$$2\alpha - 2 = 0, \quad 2\alpha = 2, \quad \boxed{\alpha = 1}$$

3. (20 points) Let L_1 and L_2 be two lines in $\mathbb{R}^3$ given by the parametric equations

$$L_1: \begin{cases} x = t + 2, \\ y = t + 3, \\ z = -3, \end{cases} \quad \text{and} \quad L_2: \begin{cases} x = s + 2, \\ y = 2s, \\ z = -s. \end{cases}$$

Determine whether the lines L_1 and L_2 are equal, parallel but not equal, skew, or intersecting. If the lines intersect, find the point of intersection.

From the parametric equations of the lines we determine the direction vectors for L_1 and L_2:

L_1 has direction vector $\vec{p} = \langle 1, 1, 0 \rangle$

L_2 has direction vector $\vec{q} = \langle 1, 2, -1 \rangle$

Vectors $\vec{p}$ and $\vec{q}$ are not parallel, therefore, L_1 and L_2 can be either skew or intersecting.

Determine whether L_1 and L_2 have a point in common. L_1 and L_2 have a point in common if the system of equations

$$\begin{cases} t + 2 = s + 2 \\ t + 3 = 2s \\ -3 = -s \end{cases}$$

has a solution.

From the last equation we have $s = 3$. Plugging this into the first two equations gives $t + 2 = 5$, $t + 3 = 6$. Both these equations are satisfied for $t = 3$. We conclude that $\boxed{L_1 \text{ and } L_2 \text{ intersect.}}$

To find the point of intersection plug $t = 3$ into the equation of L_1 (or $s = 3$ into the equation of L_2):

$$x = 5, \; y = 6, \; z = -3.$$

Therefore, $\boxed{\text{the point of intersection is } (5, 6, 3).}$

4. (20 points) Compute the distance from the point $O = (0,0,0)$ to the plane passing through the points

$$P = (1,0,0), \quad Q = (0,2,0), \quad R = (0,0,3).$$

The distance d from point O to the plane can be computed as the magnitude of the projection of the vector $\vec{PO}$ onto the normal vector $\vec{n}$ of the plane

$$d = \| \mathrm{proj}_{\vec{n}} \vec{PO} \| = \frac{|\vec{n} \cdot \vec{PO}|}{\|\vec{n}\|}$$

Compute the components of $\vec{PO}$

$$\vec{PO} = \langle -1, 0, 0 \rangle$$

As a normal vector we can take the cross product of $\vec{PQ}$ and $\vec{PR}$

$$\vec{PQ} = \langle -1, 2, 0 \rangle, \quad \vec{PR} = \langle -1, 0, 3 \rangle$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \vec{i} \cdot 6 - \vec{j}(-3) + \vec{k} \cdot 2 = \langle 6, 3, 2 \rangle$$

Compute $\vec{n} \cdot \vec{PO} = \langle 6, 3, 2 \rangle \cdot \langle -1, 0, 0 \rangle = -6$

Compute $\|\vec{n}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$

Finally, $\boxed{d = \frac{|-6|}{7} = \frac{6}{7}}$