MATH 10C (Lecture B00 - NEMISH) MIDTERM 2 11/16/22, 1-1:50pm, CENTR 115

SOLUTIONS

Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

1. (20 points) The position of an object is given by

$$\mathbf{r}(t) = \langle e^{t^2-2t+1}, \cos(\pi t), 3t^2 - 6t\rangle.$$

(a) Compute $\mathbf{v}(t)$, the velocity of the object.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle (e^{t^2-2t+1})', (\cos(\pi t))', (3t^2-6t)' \rangle$$
$$= \langle e^{t^2-2t+1}(2t-2), -\sin(\pi t)\pi, 6t-6 \rangle$$

(b) Determine the time $t_0 > 0$ at which $\mathbf{v}(t_0) = \vec{0}$.

Write $\vec{v}(t) = \vec{0}$ componentwise

$$\begin{cases} e^{t^2-2t+1}(2t-2) = 0 \\ -\sin(\pi t)\pi = 0 \\ 6t-6 = 0 \end{cases}$$

This system has a unique solution $t_0 = 1$

(c) Determine the position of the object at time t_0 (when the object's speed is zero).

The object's speed is zero only at time $t_0 = 1$.

The position of the object at $t_0 = 1$ is

$$\vec{r}(1) = \langle e^{1^2-2\cdot1+1}, \cos(\pi\cdot 1), 3\cdot 1^2 - 6\cdot 1 \rangle = \langle 1, -1, -3 \rangle$$

2. (20 points) Find the equation of the tangent plane to the surface determined by the function

$$f(x,y) = \sqrt{xy^2 + x + y}$$

at the point $(1,0)$.

The equation of the tangent plane at point (x_0, y_0) is given by $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

$f(1,0) = \sqrt{1\cdot 0^2 + 1 + 0} = \sqrt{1} = 1$

$f_x = \dfrac{y^2+1}{2\sqrt{xy^2+x+y}}$, $f_x(1,0) = \dfrac{0^2+1}{2\sqrt{1\cdot 0^2+1+0}} = \dfrac{1}{2}$

$f_y = \dfrac{2xy+1}{2\sqrt{xy^2+x+y}}$, $f_y(1,0) = \dfrac{1}{2}$

The equation of the tangent plane to f at (1,0) is

$z = 1 + \frac{1}{2}(x-1) + \frac{1}{2}(y-0)$

$z = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}y$

3. (20 points) Let

$$f(x, y) = (x + y)e^{xy^2}.$$

(a) Find the gradient of the function f at the point $(0, 0)$. Find the directional derivative of the function f at the point $(0, 0)$ in the direction $\vec{v} = \langle 1, -1\rangle$. (Hint. Do not forget to normalize $\vec{v}$.)

By definition $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0)\rangle$

$f_x = e^{xy^2} + (x+y)e^{xy^2}y^2$, $f_x(0,0) = e^{0\cdot 0^2} + (0+0)e^{0\cdot 0^2}\cdot 0^2 = 1$

$f_y = e^{xy^2} + (x+y)e^{xy^2}\cdot 2xy$, $f_y(0,0) = 1$, so $\nabla f(0,0) = \langle 1,1\rangle$

Now compute the directional derivative in the direction $\vec{v}$.

First divide $\vec{v}$ by its magnitude

$\|\vec{v}\| = \sqrt{1^2+(-1)^2} = \sqrt{2}$. Define $\vec{u} = \vec{v}/\|\vec{v}\| = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

$D_{\vec{u}} f(0,0) = \nabla f(0,0)\cdot \vec{u} = \langle 1,1\rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

(b) Find the unit vector in the direction of the maximal rate of increase for the function f at the point $(0, 0)$. What is the value of the directional derivative in this direction?

The direction of the maximal rate of increase of f at $(0,0)$ is given by $\nabla f(0,0) = \langle 1,1\rangle$.

The value of the directional derivative in this direction is equal to $\|\nabla f(0,0)\| = \|\langle 1,1\rangle\| = \sqrt{2}$

4. (20 points) Use the chain rule to find the partial derivative $\frac{\partial z}{\partial u}$ for

$$z = x^2(1+y^2)^2,$$

where

$$x = u + 2v, \quad y = uv.$$

You may leave your answer as a product of terms, but your answer should not have any derivative operations remaining to be performed. Your final answer should only be a function of u and v.

Using the chain rule we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

Now compute each term:

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial z}{\partial x} = 2x(1+y^2)^2 \qquad \frac{\partial z}{\partial y} = x^2 \cdot 2(1+y^2) \cdot 2y = 2x^2 y(1+y^2)$$

Plug the above expressions into the chain rule formula

$$\frac{\partial z}{\partial u} = 2x(1+y^2)^2 \cdot 1 + 2x^2 y(1+y^2) \cdot v$$

$$= 2x(1+y^2)(1+y^2+2xyv)$$

$$= 2(u+2v)(1+u^2v^2)(1+u^2v^2+2(u+2v)uv^2)$$

$$= 2(u+2v)(1+u^2v^2)(1+3u^2v^2+4uv^3)$$