MATH 180A (Lecture A00)

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Today: Expectation

Next: ASV 3.4

Week 5:

- Homework 3 due Friday, February 10
- Regrades of Midterm 1, HW 1, HW2 active on Gradescope until February 12, 11 PM

Expectation

Def. Let X be a discrete random variable with possible values $t_1, t_2, t_3, \dots$. The expectation or expected value or mean of X is

$$E(X) := \sum_j t_j \cdot P(X = t_j)$$

weighted average

Example Let Y be $\mathrm{Ber}(p)$.

$$E(Y) = 1 \cdot p + 0 \cdot (1-p) = p$$

Example You toss a biased coin repeatedly until the first heads. How long do you expect it to take?

N = the time the first heads comes up, $N \sim \mathrm{Geom}(p)$

$$E(N) = \frac{1}{p}$$

Examples. Binomial

$S_n \sim \mathrm{Bin}(n,p)$ $\quad (S_n = X_1 + X_2 + \cdots + X_n$ for X_j independent $\mathrm{Ber}(p))$

$$E(S_n) = \sum_{k=0}^{n} k \cdot P(S_n = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$\ell = k-1$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,\underbrace{(n-k)!}_{n-(k-1)-1}} p^k (1-p)^{n-k}$$

$$= \sum_{\ell=0}^{n-1} \frac{n!}{\ell!\,(n-1-\ell)!} p^{\ell+1} (1-p)^{n-1-\ell}$$

$$= \sum_{\ell=0}^{n-1} \frac{n \cdot (n-1)!}{\ell!\,(n-1-\ell)!} \, p \cdot p^{\ell} (1-p)^{n-1-\ell} = np \sum_{\ell=0}^{n-1} \underbrace{\binom{n-1}{\ell} p^{\ell} (1-p)^{n-1-\ell}}_{P(S_{n-1} = \ell)}$$

$\binom{n-1}{\ell}$

$$= n \cdot p$$

Notice that $E(S_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$

Examples. Poisson

$X \sim \text{Poisson}(\lambda)$

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \overset{\ell = k-1}{=} e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{\lambda^{\ell+1}}{\ell!}$$

$$= e^{-\lambda} \cdot \lambda \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda = \lambda$$

Example A factory has, on average, 3 accidents per month. Estimate the probability that there will be exactly 2 accidents this month. Take $X = \#$ accidents/month, $E(X) = 3$

$X \sim \text{Poisson}(\lambda)$, $E(X) = \lambda = 3$, $P(X=2) = \frac{\lambda^2}{2!} e^{-\lambda} = \frac{3^2}{2!} e^{-3} \approx 22.4\%$

Examples

Toss a fair coin until tails comes up. If this is on the first toss, you win 2 dollars and stop. If heads comes up, the pot doubles and you continue. That is, if the first tails is on the k-th toss, you win 2^k dollars. What is your expected winnings?

$W = 2^k$ if the first tails is on the k-th toss

$$E(W) = \sum_{k=1}^{\infty} 2^k \underbrace{P(W = 2^k)}_{\left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2}} = \sum_{k=1}^{\infty} 2^k \cdot \left(\frac{1}{2}\right)^k = \infty$$

Expectation of continuous random variables

X discrete, $X \in \{t_1, t_2, \dots\}$

$$E(X) = \sum_k t_k P(X = t_k)$$

$$= \sum_t t \cdot P(X = t)$$

X continuous $P(X=t)=0$ for each $t \in \mathbb{R}$)

with density $f_X(t)$

$$E(X) = \int_{-\infty}^{+\infty} t \cdot f_X(t)\,dt$$

Example Let $U \sim \mathrm{Unif}([a,b])$, $f_U(t) = \begin{cases} \frac{1}{b-a}, & a \le t \le b \\ 0, & \text{otherwise} \end{cases}$

$$E(U) = \int_{-\infty}^{+\infty} t \cdot f_U(t)\,dt = \int_a^b t \cdot \frac{1}{b-a}\,dt$$

$$= \frac{1}{b-a} \int_a^b t\,dt = \frac{1}{b-a} \left.\frac{t}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}$$

Example

Q: Shoot an arrow at a circular target of radius 1. What is the expected distance of the arrow from the center?

a) 1

b) $\frac{2}{3}$

c) $\frac{1}{2}$

d) $1/4$

e) 0

X = distance from center

$$F_X(r) = \begin{cases} 0, & r \leq 0 \\ r^2, & 0 < r < 1 \\ 1, & r \geq 1 \end{cases}$$

$$f_X(r) = \begin{cases} 2r, & 0 < r < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} r f_X(r)\, dr$$

$$= \int_0^1 r \cdot 2r\, dr$$

$$= 2 \cdot \frac{r^3}{3} \Big|_0^1 = \frac{2}{3}$$