MATH 180A (Lecture A00)

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Today: Expectation of a function of a random variable. Variance

Next: ASV 3.5

We'ek 6:

- Homework 4 due Friday, February 17

Expectation of continuous random variables

Example Consider function $f(t) = \begin{cases} 0, & t \le 1 \\ \frac{1}{t^2}, & t > 1 \end{cases}$

Is $f(t)$ a probability density?

$f(t) \ge 0$ $\qquad \int_{\mathbb{R}} f(t)dt = \int_1^{+\infty} \frac{1}{t^2} dt = -\frac{1}{t}\Big|_1^{+\infty} = 1$ ✓

Suppose that X is a random variable with PDF $f_X = f$.
What is $E(X)$?

$$E(X) = \int_{-\infty}^{+\infty} t f_X(t) dt = \int_1^{+\infty} t \cdot \frac{1}{t^2} dt = \int_1^{+\infty} \frac{1}{t} dt = \log t \Big|_1^{+\infty} = +\infty$$

Expectations of functions of random variables

$$\Omega \xrightarrow[\text{random variable}]{X} \mathbb{R} \xrightarrow[\text{function}]{g} \mathbb{R}$$

composition $g(X) = g \circ X$

$g(X)$ is a new random variable

Example $X \sim Bin(n,p)$ is the number of successes in n trials

$g(X) = \frac{X}{n}$ is the proportion of successful trials

$g(X) \in \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n} = 1\}$,

$$E(g(X)) = \sum_{k=0}^{n} \frac{k}{n} P(\underbrace{g(X)}_{=\frac{X}{n}} = \frac{k}{n}) = \sum_{k=0}^{n} \frac{k}{n} P(X = k) = \frac{E(X)}{n} = p$$

Proposition For a discrete random variable X

$$E(g(X)) = \sum_{s} g(s) P(X = s)$$

Note, by definition $E(g(X)) = \sum_{t} t\, P(g(X) = t)$

Expectations of functions of random variables

<u>Proposition</u> For a continuous random variable X with density f_X

$$E(g(X)) = \int_{-\infty}^{+\infty} g(t) f_X(t)\, dt$$

<u>Example</u> Let U be a uniform random variable on $[a,b]$

Then

$$E[\underbrace{U^2}_{=g(U)}] = \int_a^b t^2 \cdot f_U(t)\, dt \qquad f_U(t) = \begin{cases} \frac{1}{b-a}, & t \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

$$= \int_a^b t^2 \cdot \frac{1}{b-a}\, dt = \frac{1}{b-a} \cdot \frac{t^3}{3} \Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

Important class of functions: $g(x) = x^n$

discrete: $E(X^n) = \sum_t t^n P(X=t)$, continuous: $E(X^n) = \int_{-\infty}^{\infty} t^n f_X(t)\, dt$

moments

Expectations of functions of random variables

Example (Car accident/insurance example)

An accident causes Y dollars of damage to your car, where insurance deductible is 500 dollars.

$Y \sim \text{Unif}([100, 1500])$. What is the expected amount you pay?

X = amount you pay $= \min\{Y, 500\} = g(Y)$, $g(t) = \min(t, 500)$

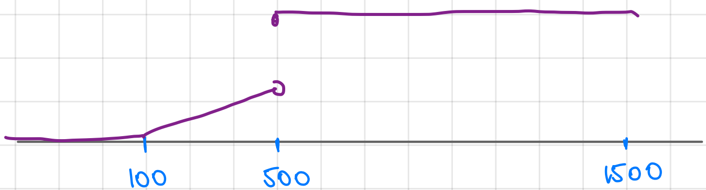
(neither discrete nor continuous)

$$E(X) = E(g(Y)) = \int_{-\infty}^{+\infty} \min\{t, 500\} \cdot f_Y(t)\, dt = \int_{100}^{1500} \min\{t, 500\} \cdot \frac{1}{1400}\, dt$$

$$= \int_{100}^{500} \frac{t}{1400}\, dt + \int_{500}^{1500} \frac{500}{1400}\, dt = \frac{1}{1400}\left(\frac{500^2}{2} - \frac{100^2}{2}\right)$$

$$\min\{t, 500\} = \begin{cases} t, & t \in [100, 500] \\ 500, & t \in [500, 1500] \end{cases}$$

$$+ \frac{500}{1400} \cdot (1500 - 500) = 442.86$$



Variance

Definition The variance of a random variable X is

$$Var(X) = E\left(\left(X - E(X)\right)^2\right)$$

- first compute $\mu = E(X)$, then compute $E(g(X))$ with $g(t) = (t-\mu)^2$
- if X is discrete, $Var(X) = \sum_t (t-\mu)^2 P(X=t)$
- if X is continuous, $Var(X) = \int_{-\infty}^{+\infty} (t-\mu)^2 f_X(t)\,dt$
- $Var(X) \geq 0$ (always)
- The square root of the variance is called

  standard deviation $\sigma(X) = \sqrt{Var(X)}$

Variance. Example

Example $X \sim Ber(p)$, $E(X) = p$

$$Var(X) = E\left((X-p)^2\right) = \sum_t (t-p)^2 P(X=t)$$

$$= (1-p)^2 \cdot p + (0-p)^2 \cdot (1-p) = p(1-p)^2 + p^2(1-p)$$

$$= p(1-p)\left(1-p+p\right) = p(1-p)$$

$$\sigma(X) = \sqrt{p(1-p)}$$

Example $U \sim Unif[a,b]$, $E(U) = \frac{a+b}{2}$

$$Var(U) = \int_{-\infty}^{+\infty} \left(t - \frac{a+b}{2}\right)^2 f_U(t)\,dt = \int_a^b \left(t - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dt = \frac{(b-a)^2}{12}$$

$$\sigma(X) = \frac{|b-a|}{\sqrt{12}}$$

Alternative formula for variance

Proposition. Let X be a random variable. Then

$$\operatorname{Var}(X) = E(X^2) - (E(X))^2$$

Proof (For continuous random variables) Let $\mu := E(X)$.

Then

$$\operatorname{Var}(X) = E((X-\mu)^2) = \int_{\mathbb{R}} (t-\mu)^2 f_X(t)\,dt$$

$$= \int_{-\infty}^{+\infty} (t^2 - 2t\mu + \mu^2) f_X(t)\,dt$$

$$= \int_{-\infty}^{+\infty} t^2 f_X(t)\,dt - 2\mu \int_{-\infty}^{+\infty} t f_X(t)\,dt + \mu^2 \int_{-\infty}^{+\infty} f_X(t)\,dt$$

$$= E(X^2) - 2\cdot\mu\cdot\mu + \mu^2\cdot 1 = E(X^2) - \mu^2$$

Example $X \sim \operatorname{Ber}(p)$, $E(X) = p$, $E(X^2) = 1^2 \cdot p + 0^2 \cdot (1-p) = p$, $\operatorname{Var}(X) = p - p^2 = p(1-p)$

Variance

Variance is a measure of how "spread out from the mean" the distribution is.

Proposition Let X be a random variable with finite expectation $E(X)=\mu$. Then

Proof $(\Leftarrow)$ Exercise

$(\Rightarrow)$ (Assume X is discrete).

$0 = Var(X) =$ $\Rightarrow$ For all t,

For all t, either or , so if

therefore,