MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Confidence intervals

Next: ASV 4.4

Week 7:

- no homework
- Midterm 2 on Wednesday, March 1 (lectures 9-18)
- Homework 5 due Friday, March 3

Summary

CLT for Bernoulli distribution:

Let $S_n \sim \mathrm{Bin}(n,p)$, let $a<b$. Then

$S_n = X_1 + \dots + X_n$, $X_j \sim \mathrm{Ber}(p)$

$Z \sim N(0,1)$

$$\lim_{n\to\infty} P\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} < b\right) = \Phi(b) - \Phi(a) = P(a < Z < b)$$

Rule of thumb

$np(1-p) > 10$

For the average $\overline{S_n} := \frac{S_n}{n}$

$$P\left(a \frac{\sqrt{p(1-p)}}{\sqrt{n}} < \overline{S_n} - p < b \frac{\sqrt{p(1-p)}}{\sqrt{n}}\right) \approx \Phi(b) - \Phi(a)$$

LLN for Bernoulli

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = 1$$

Confidence intervals. Motivation

Consider n independent trials, success rate p (unknown)

S_n = number of successes after n trials, $S_n \sim \mathrm{Bin}(n,p)$

By the LLN $\frac{S_n}{n} \to p$, $n \to \infty$ (in probability)

If n is big, then $\frac{S_n}{n}$ is close to p

random variable ← observable, estimate of p

number

Usually we do not know p, but we can get a realization of $\frac{S_n}{n}$ (flipping a coin) for finite n.

What can we say about p?

Confidence intervals. Set-up

Denote $\hat{p} := \frac{S_n}{n}$ and use the CLT for the interval $(-a, a)$

$$P\left(\frac{-a\sqrt{p(1-p)}}{\sqrt{n}} \le \hat{p} - p \le \frac{a\sqrt{p(1-p)}}{\sqrt{n}}\right) \approx 2\Phi(a) - 1$$

$$P\Big(|\hat{p} - p| \le \underbrace{\frac{a\sqrt{p(1-p)}}{\sqrt{n}}}_{=:\varepsilon}\Big) \approx 2\Phi(a) - 1 =: \gamma \text{ - confidence level}$$

We fix $\gamma \in (0,1)$ and find $a > 0$ (from $2\Phi(a) - 1 = \gamma$)

Questions:

1) For fixed n, find $\varepsilon > 0$ such that $P(|\hat{p} - p| \le \varepsilon) \ge \gamma$

2) For fixed ε, find $n \in \mathbb{N}$ such that $P(|\hat{p} - p| \le \varepsilon) \ge \gamma$

Confidence intervals

$$P\left(p \in (\hat{p}-\varepsilon, \hat{p}+\varepsilon)\right) \geq \gamma \qquad (*)$$

Notice that $\hat{p}$ is a random variable, so the interval is random

$(*)$ means that the unknown parameter p is in the interval $(\hat{p}-\varepsilon, \hat{p}+\varepsilon)$ with probability at least γ

Take some realization of $\hat{p} = \frac{S_n}{n}$ (a number), $\hat{p}_* \in \mathbb{R}$

Then $(\hat{p}_*-\varepsilon, \hat{p}_*+\varepsilon)$ is the γ-confidence interval for p

$\hat{p}_*$ is the estimate of p

We also say that $(\hat{p}-\varepsilon, \hat{p}+\varepsilon)$ is the γ-confidence interval for p

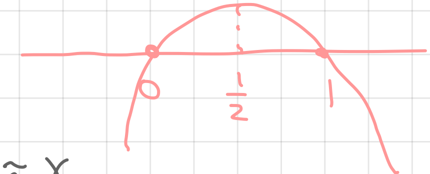
Confidence intervals. Computations

Problem: To find an estimate of p, we use

$$P\left(|\hat{p} - p| \le \underbrace{\frac{a\sqrt{p(1-p)}}{\sqrt{n}}}_{=:\varepsilon} \right) \approx 2\Phi(a) - 1 =: \gamma,$$

for which ε depends on (unknown) p.

Solution: notice that $p(1-p) \le \frac{1}{4}$



$$P\left(|\hat{p} - p| \le \underbrace{\frac{a \cdot \frac{1}{2}}{\sqrt{n}}}_{=\varepsilon} \right) \ge P\left(|\hat{p} - p| \le \frac{a\sqrt{p(1-p)}}{\sqrt{n}} \right) \approx \gamma$$

and the γ-confidence interval can be taken as $[\hat{p} - \varepsilon, \hat{p} + \varepsilon]$ with

$$\boxed{2\Phi(2\varepsilon\sqrt{n}) - 1 \ge \gamma}$$

- $\varepsilon \ge \frac{a}{2\sqrt{n}}$ for fixed n
- $\sqrt{n} \ge \frac{a}{2\varepsilon}$ for fixd ε

Confidence intervals. Example 1 (fixed n)

Flip a coin 10000 times. Number of heads is 5370.

Compute a 99%-confidence interval for $p = P(\text{Heads})$

$$n = 10000 \qquad \hat{p}_* = \frac{5370}{10000} = 0.537$$

Find (the smallest) $\varepsilon > 0$ such that

$$2 \cdot \Phi(2 \cdot \sqrt{10000} \cdot \varepsilon) - 1 \geq 0.99$$

$$\Phi(2 \cdot 100 \cdot \varepsilon) \geq 0.995$$

$$2 \cdot 100 \cdot \varepsilon \geq \Phi^{-1}(0.995) \geq 2.58$$

$$\varepsilon \geq \frac{2.58}{200}, \quad \varepsilon \geq 0.0129$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Conclusion: 99% confidence interval for p is given by

$$(0.537 - 0.0129,\ 0.537 + 0.0129)$$

Confidence intervals. Example 2 (fixed accuracy)

Flip a (possibly biased) coin. How many times should we repeat the experiment to be able to compute a 95%-confidence interval for $p = P(\text{Heads})$ of length 0.01?

$\gamma = 0.95$, $\varepsilon = 0.005$

$$2\Phi(2\cdot\sqrt{n}\cdot\varepsilon) - 1$$

$$= 2\cdot\Phi(2\cdot\sqrt{n}\cdot 0.005) - 1 \geq 0.95$$

$$\Phi(0.01\sqrt{n}) \geq 0.975$$

$$0.01\cdot\sqrt{n} \geq \Phi^{-1}(0.975) = 1.96$$

$$\sqrt{n} \geq 196, \quad n \geq (196)^2$$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Conclusion: if we toss the coin $(196)^2$ times and compute the proportion of heads $\hat{p}_*$, then $(\hat{p}_* - 0.005, \hat{p}_* + 0.005)$ is the 95%-conf. int.

Confidence intervals. Polling

Without going into details (see example 4.14, Remark 4.16)
Part of the population (unknown p) prefers product A / supports candidate B / etc... . We interview n individuals and k of them give a positive answer (about product/candidate)
What can we say about p?

Fix confidence level γ (often $\gamma = 0.95$).

Again, $\hat{p}_* = \frac{k}{n}$ is the estimate of p, and
$\left(\frac{k}{n} - \varepsilon, \frac{k}{n} + \varepsilon\right)$ gives a $\left(2\Phi(2\varepsilon\sqrt{n}) - 1\right)$ - confidence interval,

so from $2\Phi(2\varepsilon\sqrt{n}) - 1 \geq \gamma$ we find ε for given n, or find n for given ε (how many people we have to interview to achieve the derised margin error)

Confidence intervals. Example (rock vs rap)

You ask 400 randomly chosen San Diegans whether they prefer rock or rap. 230 choose rock. Give a 99% confidence interval for the part of the population that prefers rock.

$$n = 400, \quad \gamma = 0.99, \quad \hat{p}_* = \frac{230}{400} = 0.575$$

Find $\varepsilon > 0$ such that $2\Phi(2 \cdot \varepsilon \cdot \sqrt{400}) - 1 \geq 0.99$

$$\Phi(2 \cdot \varepsilon \cdot 20) > 0.995$$

$$40 \cdot \varepsilon \geq \Phi^{-1}(0.995) = 2.58$$

$$\varepsilon \geq \frac{2.58}{40} = 0.0645$$

99% confidence interval is

$$(0.575 - 0.0645,\ 0.575 + 0.0645)$$