MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Poisson approximation

Next: ASV 5.1

Week 8:

- Midterm 2 on Wednesday, March 1 (lectures ~~8~~ 9-18)
- Homework 5 due Friday, March 3

CLT for Bernoulli / Binomial

Let $S_n \sim \mathrm{Bin}(n,p)$, $a<b$. Then

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) = \Phi(b) - \Phi(a)$$

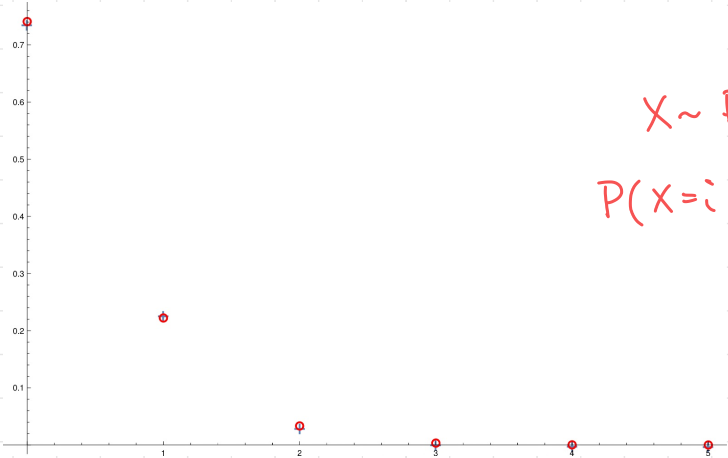
- n independent trials
- probability of success p
- rule of thumb: $np(1-p) > 10$

If this holds, then $N(np, np(1-p))$ gives a good approximation of $\mathrm{Bin}(n,p)$

What about other regimes? $np(1-p) < 10$

Example

$S_{10} \sim \mathrm{Bin}(10, 0.03)$

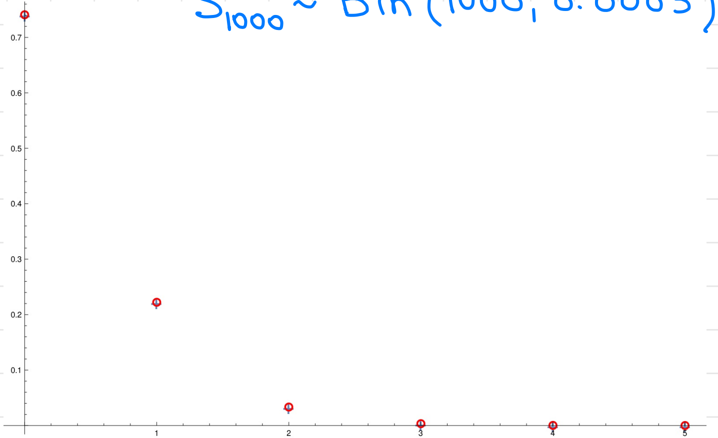


$P(S_{10} = i)$

$X \sim \mathrm{Poisson}(0.3)$

$P(X = i)$

$S_{1000} \sim \mathrm{Bin}(1000, 0.0003)$



$P(S_{1000} = i)$

Poisson approximation

$\lambda > 0$, $X \sim \text{Poisson}(\lambda)$, for $k = 0, 1, 2, \dots$

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Lecture 13: $E(X) = \lambda$

$\mathrm{Var}(X) = \lambda$ (similarly as $E(X)$)

Lecture 12: Let $\lambda > 0$ and n positive integer.

Let $S_n \sim \mathrm{Bin}(n, \frac{\lambda}{n})$ (assume $\frac{\lambda}{n} < 1$).

Then for any fixed $k \in \{0, 1, 2, \dots\}$

$$\lim_{n \to \infty} P(S_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson approximation for finite n

When to approximate by Poisson?

<u>Proposition</u> Let $X \sim \text{Bin}(n,p)$ and $Y \sim \text{Poisson}(np)$.
Then for any $A \subset \{0,1,2,\dots\}$

$$|P(X\in A) - P(Y\in A)| \le np^2$$

for normal

$$\| \quad | \le \frac{1}{\sqrt{np(1-p)}}$$

If np^2 is small → use Poisson approximation.

<u>Remark.</u> If $p = \frac{\lambda}{n}$, then one can have $np(1-p) = \lambda(1-\frac{\lambda}{n}) \approx 10$
and at the same time $np^2 = \frac{\lambda^2}{n} = \frac{1}{100}$ $\quad n = 10^4$

Then both approximations, normal by $N(\lambda,\lambda)$ and Poisson (λ), are close to $\text{Bin}(n,\frac{\lambda}{n})$

Example. Approximating Bin (n,p)

John flips a coin repeatedly until the tails comes up and counts the number of flips.

Approximate the probability that in a year there are at least 3 days when he needs more that 10 flips.

Let X = number of coin flips on a given day, $X \sim \text{Geom}(\frac{1}{2})$

$n = 365$ $\quad p = P(X > 10) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$, $\quad np = \frac{365}{1024} = 0.356$

If Y = number of days with more than 10 flips, then $Y \sim \text{Bin}\left(365, \frac{1}{1024}\right)$, $np^2 = 365 \cdot 2^{-20}$ is small, so we can use the Poisson approximation, $Z \sim \text{Pois}(0.356)$,

$$P(Y \geq 3) \approx P(Z \geq 3) = 1 - P(Z=0) - P(Z=1) - P(Z=2)$$

$$= 1 - e^{-0.356}\left(1 + 0.356 + \frac{(0.356)^2}{2}\right) \approx 0.00577$$

Approximating probabilities of rare events

Poisson distribution is used to model the occurances of rare events. Examples:

customers arriving in a store ⟷ all potential customers decide independently to come or not

number of emergency calls ⟷ all people in the city have an emergency or not "independently" of each other

number of car accidents ⟷ all drivers in the county have accidents (or not) "independently"

number of goals scored in a hockey game ⟷ a lot of "independent" shots

Example

Number of phone calls in a day can be modeled by Poisson random variable. We know that on average 0.5% of the time the call center receives no calls at all. What is the average number of calls per day?

$X \sim \text{Poisson}(\lambda)$, $E(X) = \lambda - ?$

It is given that $P(X=0) = 0.005 = \frac{1}{200} = e^{-\lambda}$

Therefore, $\log e^{-\lambda} = -\lambda = \log \frac{1}{200} = -\log 200$

$$\lambda = \log 200 \approx 5.298$$

Exercise

10% of households earn > 80000 \$

0.25% of households earn > 450000 \$

Choose 400 households at random. Denote

$X = \#$ households > 80000, $Y = \#$ households > 450000 \$

Estimate $P(X \geq 48)$ and $P(Y \geq 2)$

Normal approx. ok

1) For X: $n = 400$, $p_X = 0.1$, $np_X = 40$ $np_X(1-p_X) = 36 > 10$

$$P(X \geq 48) = P\left(\frac{X-40}{\sqrt{36}} \geq \frac{48-40}{\sqrt{36}}\right) \approx 1 - \Phi\left(\frac{8}{6}\right) \approx 0.1056$$

2) For Y, $n = 400$, $p_Y = \frac{1}{400}$, $np_Y = 1$, $np_Y^2 = \frac{1}{400}$ → use Poisson approximation

$$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1) \approx 1 - e^{-1} - e^{-1} \approx 0.2642$$