MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Definition of probability

Next: ASV 1.2

Week 1:

- check the course website
- homework 1 (due Friday, January 20)
- join Piazza

Probability theory

The goal of probability theory is to build mathematical models of experiments with random outcomes

Random outcome = impossible to be predicted with certainty

1654: starting point, mathematical treatment of gambling problems (Fermat, Pascal)

1933: modern rigorous foundation of probability theory (Kolmogorov)

Warm-up problem

What is the probability that there are at least two student in this room having birthday on the same day (MM/DD)?

72 students, 365 possible birthday dates

(a) $< 0.1\%$

(b) $\approx 4\% \approx \binom{72}{2} / \binom{365}{2}$

(c) $\approx 20\% \approx \frac{72}{365}$

(d) $> 99.9\%$

Moral: Intuition may be misleading

Axioms of probability

How to construct a mathematical model of an experiment with random outcome?

Def. Probability space is the triple $(\Omega, \mathcal{F}, P)$, where

- Ω is the set of all possible outcomes of the experiment ; we call it the sample space
- $\mathcal{F}$ is a collection of subsets of Ω (events)
- P is a function that assigns to each event a real number and satisfies the following properties:

AXIOMS OF PROBABILITY

(i) $0 \le P(A) \le 1$ for all $A \in \mathcal{F}$

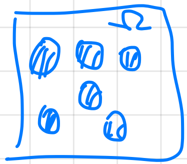
(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$

(iii) If $A_1, A_2, A_3, \cdots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

$A \cup B$

$A \cap B$, AB



Examples

We call function P that satisfies properties (i)-(iii) a probability measure, or simply probability.

Example 1: Tossing a coin.

$$\Omega = \{H, T\}, \quad \mathcal{F} = \{\emptyset, \{H\}, \{T\}, \underbrace{\{H, T\}}_{=\Omega}\}^{\Omega}, \quad P(\emptyset) = 0, \quad P(\Omega) = 1$$

$$P(\{H\}) = \alpha, \quad P(\{T\}) = \beta = 1 - \alpha$$

$\{H\}$ and $\{T\}$ are disjoint, therefore, by (iii)

$$\alpha + \beta = P(\{H\}) + P(\{T\}) \overset{(iii)}{=} P(\{H\} \cup \{T\}) = P(\Omega) = 1$$

For any $\alpha \in [0,1]$ we have a different probability measure on Ω and $\mathcal{F}$.

Fair coin: $\alpha = \frac{1}{2}$, $P(\{H\}) = P(\{T\}) = \frac{1}{2}$

Examples

Example 2: rolling a fair die

$\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\text{all subsets of } \Omega\}$ $|\mathcal{F}| = 2^6$

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = \frac{1}{6}$$

What about the events? Take $A = \{2, 4, 6\} \subset \Omega$.

$$P(A) = P(\{2\} \cup \{4\} \cup \{6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$B = \{2, 3, 5\}$: $P(B) =$

$C = \{3, 6\}$: $P(C) =$

A = "even number"

B = "prime number"

C = "divisible by 3"

$P(A \cup B) =$

$P(B \cap C) =$