MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Definition of probability.
Random sampling
Next: ASV 1.3-1.4

Week 1:

- check the course website
- homework 1 (due January 20)
- join Piazza
- Canvas Quizzes → First day survey

Axioms of probability

Mathematical model of an experiment with random outcome

<u>Def.</u> Probability space is the triple $(\Omega, \mathcal{F}, P)$, where

- Ω is the set of all possible outcomes of the experiment; we call it the sample space
- $\mathcal{F}$ is a collection of subsets of Ω (events)
- P is a function that assigns to each event a real number and satisfies the following properties:

AXIOMS OF PROB

(i) $0 \le P(A) \le 1$ for all $A \in \mathcal{F}$

(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$

$A_i \cap A_j = \emptyset$

(iii) If $A_1, A_2, A_3, \dots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

P is the probability measure (or simply probability)

Examples

Example 2: rolling a fair die

$\Omega = \{1,2,3,4,5,6\}$, $\mathcal{F} = \{\text{all subsets of } \Omega\}$

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = \frac{1}{6}$$

What about the events? Take $A = \underbrace{\{2,4,6\}}_{}\subset \Omega$. $\quad = \{2\}\cup\{4\}\cup\{6\}$

$$P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

$B = \{2,3,5\}$: $P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$C = \{3,6\}$: $P(C) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

A = "even number"

B = "prime number"

C = "divisible by 3"

$$P(A \cup B) = P(\{2,3,4,5,6\}) = \frac{5}{6}$$

$$P(B \cap C) = P(\{3\}) = \frac{1}{6}$$

Repeated experiments

What is the sample space if we toss the coin twice?

The outcome is a pair (X, Y) with $X, Y \in \{H, T\}$

The collection of such pairs is called the Cartesian product of $\{H,T\}$ and $\{H,T\}$, denoted $\underset{\Omega \times \Omega}{\{H,T\} \times \{H,T\}}$

$$\{H,T\}^2 = \{H,T\} \times \{H,T\} = \{(H,H), (H,T), (T,H), (T,T)\} \leftarrow \text{sample space}$$

More generally, for any sets $\Omega_1, \Omega_2, \dots, \Omega_k$

$$\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k = \{(\omega_1, \omega_2, \dots, \omega_k) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2, \dots, \omega_k \in \Omega_k\}$$

sample space if we perform experiment 1 with s.s. Ω_1

experiment 2 with s.s. Ω_2

⋮

experiment k with s.s. Ω_k

$$\Omega^k = \underbrace{\Omega \times \Omega \times \cdots \times \Omega}_{k \text{ times}}$$

Finite sample space

Consider a special case when $\# \Omega < \infty$. Then

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\} \quad \text{for } n = \# \Omega$$

Any event $A \subset \Omega$ is a finite union of $\{\omega_i\}$.
The singleton sets $\{\omega_1\}, \dots, \{\omega_n\}$ are disjoint.

Therefore, if $A = \{a_1, \dots, a_k\}$ for some $a_i \in \Omega$, then

$$P(A) = P(\{a_1\}) + P(\{a_2\}) + \dots + P(\{a_k\})$$

What if additionally we have that $P(\{\omega_1\}) = \dots = P(\{\omega_n\})$?
↳ random sampling

Uniform probability measure and random sampling

If Ω is finite, the uniform probability measure is defined by the following property:

$\#\Omega < \infty$
$\| $
n

$$\text{for each } \omega \in \Omega, \quad P(\{\omega\}) = \frac{1}{\#\Omega}$$

From (*) this implies that

$$\text{for any event } A, \quad P(A) = \frac{\#A}{\#\Omega}$$

This means that for such models calculating probabilities is reduced to counting.

Example Roll a fair die twice. What is the probability that the sum is 4? $\Omega = \{(i,j) : 1 \le i,j \le 6\}$, $\#\Omega = 36$

$A = \{(1,3),(2,2),(3,1)\}$, $\#A = 3$ $\quad P(A) = \frac{3}{36} = \frac{1}{12} \approx 8.3\%$

Uniform probability measure and random sampling

Example A fair coin is tossed 3 times.

$A = \{\text{at least two tails}\}$

$B = \{\text{exactly two tails}\}$

$\Omega = \{ (x_1, x_2, x_3) : x_i \in \{H, T\} \}$, $\#\Omega = 2^3 = 8$

$A = \{TTT, TTH, THT, HTT\}$, $\#A = 4$

$B = \{TTH, THT, HTT\}$, $\#B = 3$

$P(A) = \frac{1}{2}$, $P(B) = \frac{3}{8}$

Warm-up exercise

There are 10 people on a committee.

How many different ways are there to select a subcommittee of 4 people?

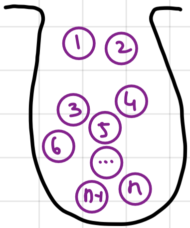
(a) $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\,000$

(b) $10 \cdot 9 \cdot 8 \cdot 7 = 5040$

(c) $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$

(d) $\frac{10!}{4!} = 151\,200$

Combinatorics



A collection of n labelled balls $\{1, 2, 3, \dots, n\}$ are in an urn. k are taken out one by one.

Q: How many ways?

Possible scenarios:

Replacement

- with replacement
- without replacement

Order

- order balls come out matters
- order does not matter

$n = 5$, $k = 3$ (choose 3 balls)	order matters	order doesn't matter
with replacement	1 2 1 $\neq$ 1 1 2	1 2 1 $=$ 1 1 2
without replacement	1 2 3 $\neq$ 3 2 1	1 2 3 $=$ 3 2 1

Combinatorics

Sampling with replacement, order matters

sample space : $\Omega = \{ (b_1, \dots, b_k) : 1 \le b_i \le n \} = \{1, \dots, n\}^k$

Sampling without replacement, order matters

sample space : $\Omega = \{ (b_1, \dots, b_k) : 1 \le b_i \le n,\ b_i \ne b_j \text{ if } i \ne j \}$

Sampling without replacement, order does not matter

sample space : $\Omega = \{ \{b_1, \dots, b_k\},\ 1 \le b_i \le n,\ b_i \ne b_j \text{ if } i \ne j \}$

	order matters	order doesn't matter
with replacement	$\#\Omega =$	
without replacement	$\#\Omega =$ $=$	$\#\Omega =$ $=$