# MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

## Today: Definition of probability. Random sampling

## Next: ASV 2.1

Week 1:

- check the course website
- homework 1 (due Friday, January 20)
- join Piazza

# Last time

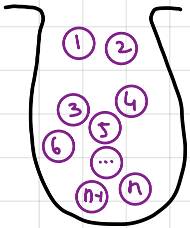
If $\Omega$ is finite, the uniform probability measure is defined by the following property:

$$\text{for each } \omega \in \Omega, \quad P(\{\omega\}) = \frac{1}{\#\Omega}$$

From (*) this implies that

$$\text{for any event } A, \quad P(A) = \frac{\#A}{\#\Omega}$$

# Combinatorics



A collection of $n$ labelled balls $\{1, 2, 3, \ldots, n\}$ are in an urn. $k$ are taken out one by one.

Q: How many ways?

Possible scenarios:

**Replacement**

- with replacement
- without replacement

**Order**

- order balls come out matters
- order does not matter

| $n=5$, $k=3$ (choose 3 balls) | order matters | order doesn't matter |
|---|---|---|
| with replacement | 1 2 1 $\neq$ 1 1 2<br>$(b_1, b_2, b_3)$ | 1 2 1 $=$ 1 1 2 |
| without replacement | 1 2 3 $\neq$ 3 2 1<br>$(b_1, b_2, b_3)$, $b_i \neq b_j$ if $i \neq j$ | 1 2 3 $=$ 3 2 1<br>$\{b_1, b_2, b_3\}$ |

# Combinatorics

Sampling with replacement, order matters

$$\Omega = \{(b_1,\dots,b_k) : 1 \le b_i \le n\} = \{1,\dots,n\}^k$$

Sampling without replacement, order matters

$$\Omega = \{(b_1,\dots,b_k) : 1 \le b_j \le n,\ b_i \neq b_j \text{ if } i \neq j\}$$

Sampling without replacement, order does not matter

$$\Omega = \{\{b_1,\dots,b_k\} : 1 \le b_i \le n,\ b_i \neq b_j \text{ if } i \neq j\}$$

| | order matters | order doesn't matter |
|---|---|---|
| with replacement | $\#\Omega =$ | |
| without replacement | $\#\Omega =$<br>$=$ | $\#\Omega =$<br>$=$ |

# Important remark. Examples

Each of these three models leads to a **uniform probability measure!** on the corresponding sample space

Example (sampling with replacement)

Toss a fair coin $n$ times; record a statistic observing

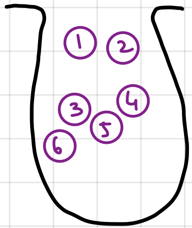
$\# H$ vs $\# T$

Take $n = 10$. Q: compute $P$(odd rolls are all $H$)

$\Omega = \{ (c_1, c_2, \ldots, c_{10}) : c_j \in \{H, T\} \}$, $\#\Omega =$

# Examples

Example (Sampling without replacement, order matters)




There are 6 labelled balls in an urn.

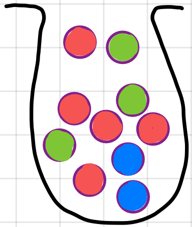
3 are removed in sequence (without replacement) and lined up in order.

Q: What is the probability that the first two are (4,5)?

$$\Omega = \{ (b_1, b_2, b_3) : 1 \leq b_j \leq 6, \ b_1 \neq b_2, \ b_2 \neq b_3, \ b_1 \neq b_3 \}$$

# Examples

**Example** (Sampling without replacement, order does not matter)



An urn contains 10 balls: $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$

two blue, three green, five red

3 balls are chosen without replacement.

Q: Compute P( choose 2 green and one red )

$\# \Omega =$

# Combinatorics

- selecting $k$ objects among $n$, with replacement

$$\# \text{ways} = n^k$$

- selecting $k$ objects among $n$, without replacement
  order matters

$$\# \text{ways} = n(n-1)(n-2)\cdots(n-k+1) \qquad (k \leq n)$$

- selecting $k$ objects among $n$, without replacement
  order doesn't matter

$$\# \text{ways} = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

- # of ways to order $n$ objects : $n(n-1)\cdots 1 = n!$

# Warm-up exercise

There are 10 people on a committee.

How many different ways are there to select a subcommittee of 4 people?

(a) $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\,000$

(b) $10 \cdot 9 \cdot 8 \cdot 7 = 5040$

(c) $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$

(d) $\frac{10!}{4!} = 151\,200$

## Example

You have a deck of 52 cards (4 suits × 13 ranks).

You choose 5 cards uniformly at random.

What is the probability that you choose 3 cards of one rank + 2 cards of another rank (full house)?

$\Omega =$ , $\#\Omega = \binom{52}{5}$

$A =$

$\#A =$

$P(\text{full house}) =$

# Infinite sample space

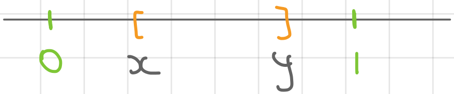
If $\# \Omega = \infty$, then we need a different notion of uniform probability measure.

**Example** A random number is chosen in $[0,1]$.

(a) What is the probability that it is $\geq 0.6$

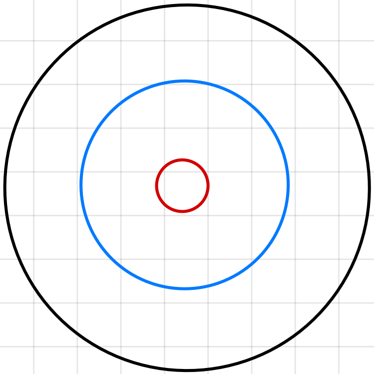
(b) What is the probability that it is $= \frac{1}{2}$

$(\Omega, \mathcal{F}, P)$ :




If $\Omega = [a,b]$, then take

# Infinite sample space



An archery target is a disk 50 cm in diameter

A blue disk is 25 cm in diameter

A red disk is 5 cm in diameter

Given that you hit the target (randomly), what are the chances of hitting the blue disk? The red disk?

$\Omega$ = target , $P(A) =$ , $\mathcal{F} =$

General rule :

# Decompositions

**Example** A fair die is rolled 5 times. What is the probability that you get one at least 3 times?

$$A = \{\text{at least 3 ones}\} =$$

$P(A) =$

Consider $A_3$, exactly 3 ones. How many ways?

$\square \quad \square \quad \square \quad \square \quad \square$

1)

2)

# Decompositions

Example A fair die is rolled 5 times. What is the probability that you get at least one double?

$A = \{$ some number comes up at least twice $\}$

$A_k = \{ k$ comes up at least two times $\}$

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

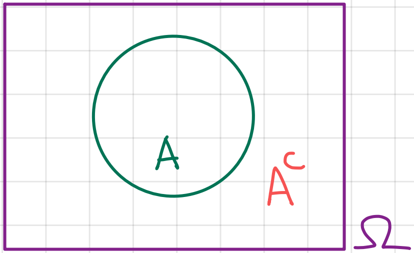
We can further decompose

$$A_1 = A_1^2 \cup A_1^3 \cup A_1^4 \cup A_1^5 \cup A_1^6 \quad , \quad A_1^j = \{ 1 \text{ comes up exactly } j \text{ times} \}$$

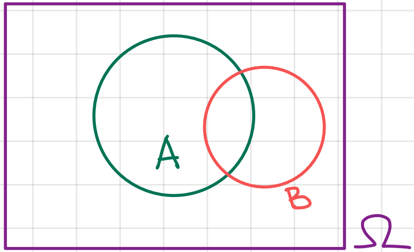
$$\vdots$$

Easy solution:

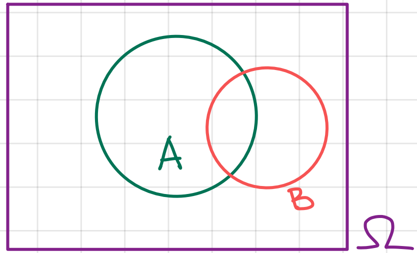
# Intersections and unions




$A \cup B$



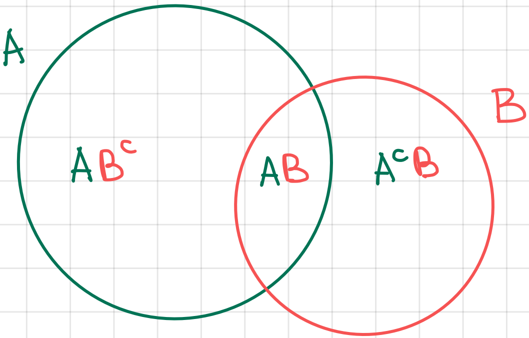

$A \cap B$




Sometimes we have to take intersections into account.

Notation: $A \cup B = \{$all outcomes in either A or B or both$\}$

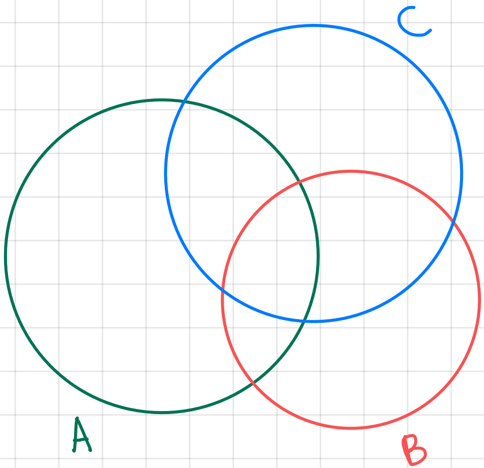
$A \cap B = \{$all outcomes in both A and B$\} = AB$




$$A \cup B = AB^c \cup AB \cup A^cB$$

# Principle of inclusion / exclusion

The probability of a union can be computed by adding the probabilities, then subtracting off the intersection overcounted. If you have more sets, you have to keep going and re-add back in over-subtracted pieces etc...



# Principle of inclusion / exclusion

**Example** Among students enrolled in MATH 180A

60 % are Math majors

20 % are Physics majors

5 % are majoring in both Math and Physics

A student is chosen randomly from the class. What is the probability that this student is neither a Math major nor a Physics major?

$A = \{\text{Math}\}$

$B = \{\text{Physics}\}$

$C = \{\text{neither}\}$