MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Consequences of the axioms of probability

Next: ASV 1.3-1.4

Week 2:

- homework 1 (due Friday, January 20)

Infinite sample space

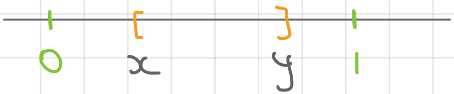
If $\# \Omega = \infty$, then we need a different notion of uniform probability measure.

Example A random number is chosen in $[0,1]$.

(a) What is the probability that it is ≥ 0.6

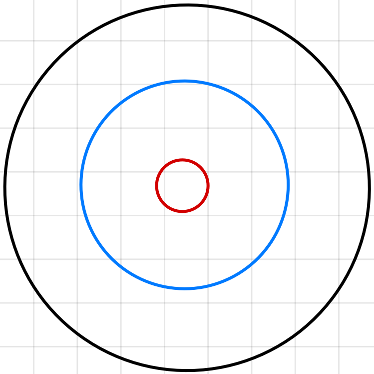
(b) What is the probability that it is $= \frac{1}{2}$

$(\Omega, \mathcal{F}, P)$:



If $\Omega = [a,b]$, then take

Infinite sample space



An archery target is a disk 50 cm in diameter

A blue disk is 25 cm in diameter

A red disk is 5 cm in diameter

Given that you hit the target (randomly), what are the chances of hitting the blue disk? The red disk?

Ω = target , $P(A) =$, $\mathcal{F} =$

General rule :

Decompositions

Example A fair die is rolled 5 times. What is the probability that you get one at least 3 times?

$A = \{\text{at least 3 ones}\} =$

$P(A) =$

Consider A_3, exactly 3 ones. How many ways?

□ □ □ □ □

1)

2)

Decompositions

<u>Example</u> A fair die is rolled 5 times. What is the probability that you get at least one double?

$A = \{$ some number comes up at least twice $\}$

$A_k = \{ k$ comes up at least two times $\}$

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

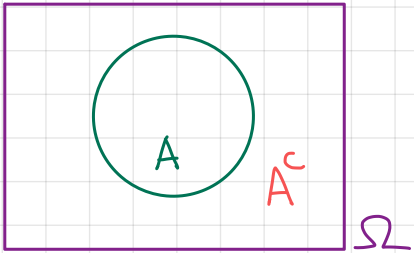
We can further decompose

$A_1 = A_1^2 \cup A_1^3 \cup A_1^4 \cup A_1^5 \cup A_1^6$, $A_1^j = \{ 1$ comes up exactly j times$\}$

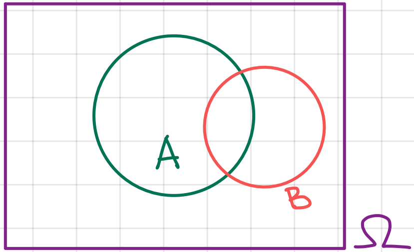
$\vdots$

Easy solution:

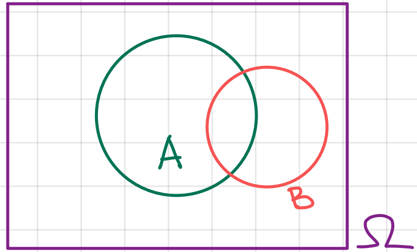
Intersections and unions



A∪B



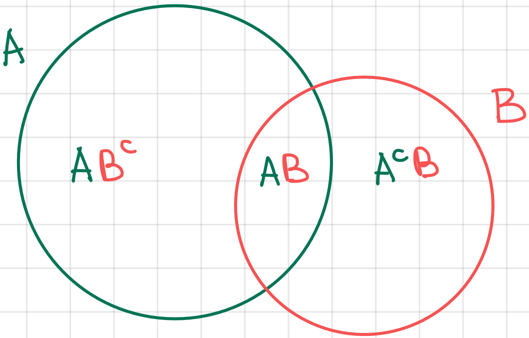
A∩B



Sometimes we have to take intersections into account.

Notation: $A \cup B = \{\text{all outcomes in either } A \text{ or } B \text{ or both}\}$

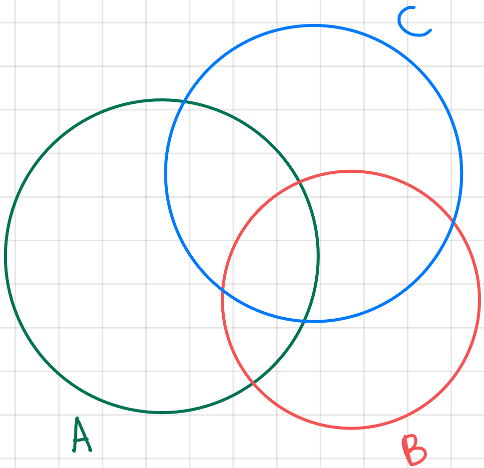
$A \cap B = \{\text{all outcomes in both } A \text{ and } B\} = AB$



$$A \cup B = AB^c \cup AB \cup A^cB$$

Principle of inclusion / exclusion

The probability of a union can be computed by adding the probabilities, then subtracting off the intersection overcounted. If you have more sets, you have to keep going and re-add back in over-subtracted pieces etc...



Principle of inclusion / exclusion

Example Among students enrolled in MATH 180A

60 % are Math majors

20 % are Physics majors

5 % are majoring in both Math and Physics

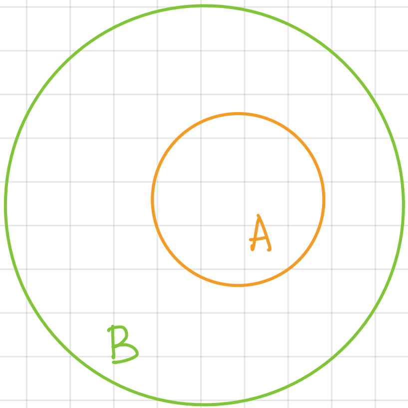
A student is chosen randomly from the class. What is the probability that this student is neither a Math major nor a Physics major?

$A = \{ \text{Math} \}$

$B = \{ \text{Physics} \}$

$C = \{ \text{neither} \}$

Monotonicity



If $A \subseteq B$, then

Indeed, $P(B) =$

$\geq$

In particular, $P(A \cup B) \geq \max\{P(A), P(B)\}$

$P(A \cap B) \leq \min\{P(A), P(B)\}$

Useful tools

- $P(A) + P(A^c) = 1$ (events and their complements)
- $A \subset B$ implies $P(A) \leq P(B)$ (monotonicity)
- $P(A \cup B) = P(A) + P(B) - P(AB)$ (inclusion - exclusion)

Conditional probability

Example 1. Your friend rolls two fair dice. What is the probability the sum is 10.

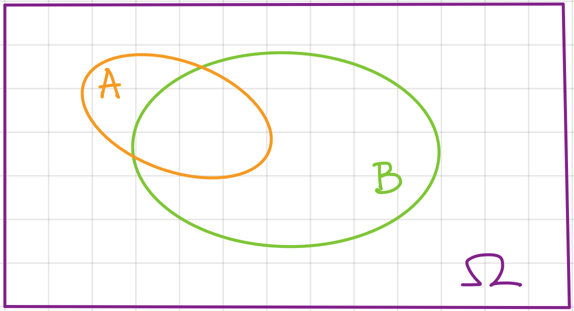
$\Omega =$

$A =$

$P(A) =$

Example 2 Your friend rolls two fair dice and tells you that the sum that came up is a two digit number. What is the probability that the sum is 10?

Conditional probability



If we know that the event B happened

- keep the same Ω and $\mathcal{F}$
- define new probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ that takes into account the additional information

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<u>Def</u>. Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$ the conditional probability of A given B is defined as

Conditional probability

If $P(B) > 0$, then the conditional probability $P(\cdot \mid B)$ satisfies all the properties of probability:

axioms of probability, probability of the complement, monotonicity, inclusion-exclusion ...

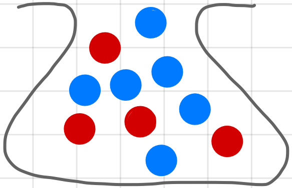
Remark If Ω is finite and P is uniform, then

$$P(A \mid B) =$$

Example Roll two fair dice. $A = \{\text{sum is } 10\}$,

$B = \{\text{sum is a two-digit number}\}$

Examples



An urn contains 4 red ball and 6 blue balls. Three balls are sampled without replacement. What is the probability that exactly two are red?

Suppose we know a priori that at least one red ball is sampled. What is the conditional probability of A?

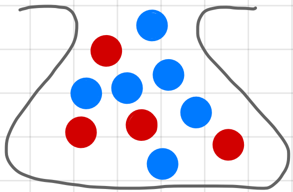
Multiplication rule

By definition $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow$

For $A \cap B \cap C$: $P(A \cap B \cap C) =$

Example



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