MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Conditional probability. Independence

Next: ASV 3.1

Week 3:

- homework 1 (due Monday, January 23)

Conditional probability

Def. Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$ the conditional probability of A given B is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

If $P(B) > 0$, then the conditional probability $P(\cdot \mid B)$ satisfies all the properties of probability:
axioms of probability, probability of the complement, monotonicity, inclusion-exclusion ...

Remark If Ω is finite and P is uniform, then

$$P(A \mid B) = \frac{\# A \cap B}{\# B}$$

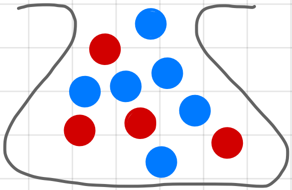
Multiplication rule

By definition $P(B|A) = \dfrac{P(A\cap B)}{P(A)}$

$$\Rightarrow P(A\cap B) = P(A)\cdot P(B|A) = P(B)P(A|B) \leftarrow \text{multiplication rule}$$

For $A\cap B\cap C$: $P(A\cap B\cap C) = P(AB)P(C|AB) = P(A)P(B|A)P(C|AB)$

Example



An urn contains 4 red balls and 6 blue balls. Two balls are sampled without replacement. What is the probability that both are red?

$B_1 = \{1^{st}$ ball is red$\}$

$B_2 = \{2^{nd}$ ball is red$\}$

$$P(B_1\cap B_2) = P(B_1)\cdot P(B_2|B_1) = \frac{4}{10}\cdot\frac{3}{9} = \frac{2}{15}$$

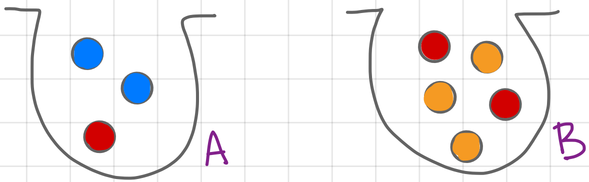
$$P(B_1) = \frac{4}{10}, \quad P(B_2|B_1) = \frac{3}{9}$$

$$P(B_1\cap B_2) = \frac{\binom{4}{2}}{\binom{10}{2}}$$

Two-stage experiments

- perform an experiment, measure a random outcome
- perform a second experiment whose setup depends on the outcome of the first!

Example



- first choose an urn at random
- then sample a ball at random from the chosen urn

Q: What is the probability that the sampled ball is red?

$R = \{\text{sample red ball}\}$, $A = \{\text{choose urn } A\}$, $B = \{\text{choose urn } B\}$

$$P(R) = P((R \cap A) \cup (R \cap B)) = P(R \cap A) + P(R \cap B)$$

$$= P(A)\,P(R \mid A) + P(B)\,P(R \mid B)$$

$$= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{5} = \frac{11}{30}$$

Law of total probability

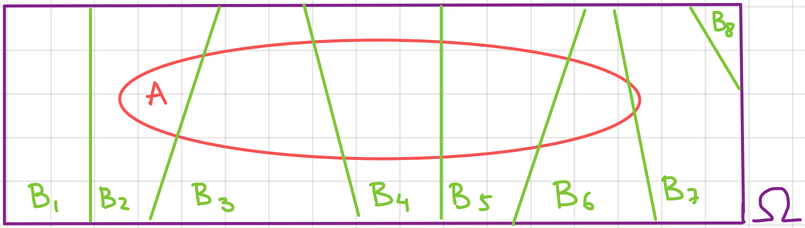
Let $B_1, B_2, \dots, B_n$ be a partition of Ω

(i.e., B_i are disjoint, $B_1 \cup B_2 \cup \dots \cup B_n = \Omega$, $P(B_i) > 0$).

Then for every event A:

$$P(A) = P(AB_1 \cup AB_2 \cup \dots \cup AB_n) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$= \sum_{i=1}^{n} P(B_i)\,P(A \mid B_i)$$



<u>Example</u> 90% of coins are fair, 9% are biased to come up heads 60% of times, 1% are biased to come up heads 80%. You find a coin on the street.

How likely is it to come up heads?

Law of total probability

Define $A = \{\text{coin comes up heads}\}$, $B_1 = \{\text{coin is fair}\}$

$B_2 = \{\text{coin is 60\% biased}\}$ $B_3 = \{\text{coin is 80\% biased}\}$

- B_1, B_2, B_3 form a partition and

  $P(B_1) = 0.9$, $P(B_2) = 0.09$, $P(B_3) = 0.01$

- $P(A|B_1) = 0.5$ $P(A|B_2) = 0.6$ $P(A|B_3) = 0.8$

Then using the law of total probability

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$

$$= 0.9 \cdot 0.5 + 0.09 \cdot 0.6 + 0.01 \cdot 0.8 = 0.512$$

Another question: In the same setting, you find a coin and toss it. It comes up heads. How likely is it that this coin is 80% biased (heavily biased)?

Important remark

We know that $P(A|B_3) = 0.8$.

What can we say about $P(B_3|A)$?

Generally speaking, $P(A|B) \neq P(B|A)$

Example According to Forbes, there are 2668 billionaires in the world, 2357 of them are men.

$$P(M|B) = \frac{2357}{2668} \approx 88\% \neq P(B|M)$$

Example Prosecutor's fallacy:

$E = \{\text{evidence on the defendant}\}$

$I = \{\text{defendant is innocent}\}$

Althoug $P(E|I)$ is usually small, $P(I|E)$ may be much higher

Bayes' Rule (relation between $P(A|B)$ and $P(B|A)$)

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(B)\,P(A|B)}{P(A)}$$

This formula is often used with the law of total probability.

Let $B_1, B_2, \dots, B_n$ be a partition of the sample space. Then for any event A with $P(A)>0$

$$P(B_k|A) = \frac{P(B_k)\,P(A|B_k)}{P(A)} = \frac{P(B_k)\,P(A|B_k)}{\sum_{i=1}^{n} P(B_i)\,P(A|B_i)}$$

Example. $A=\{\text{coin comes up heads}\}$, $B_1=\{\text{coin is fair}\}$

$B_2=\{\text{coin is } 60\% \text{ biased}\}$ $B_3=\{\text{coin is } 80\% \text{ biased}\}$

Given: $P(B_1)=0.9$, $P(B_2)=0.09$, $P(B_3)=0.01$ $P(A|B_1)=0.5$, $P(A|B_2)=0.6$, $P(A|B_3)=0.8$

We have computed that $P(A)=\sum_{i=1}^{3} P(B_i)P(A|B_i)=0.512$

Then $P(B_3|A) = \dfrac{0.01\cdot 0.8}{0.512} \approx 0.0156$

Bayes' rule

Example

Suppose that a certain test (e.g., virus X test) is 99% accurate (1% false positives, 1% false negatives). 0.25% of the population have this virus.
You test positive. What is the probability you have this virus?

(a) 99%

(b) 20%

(c) 1%

(d) 0.3%

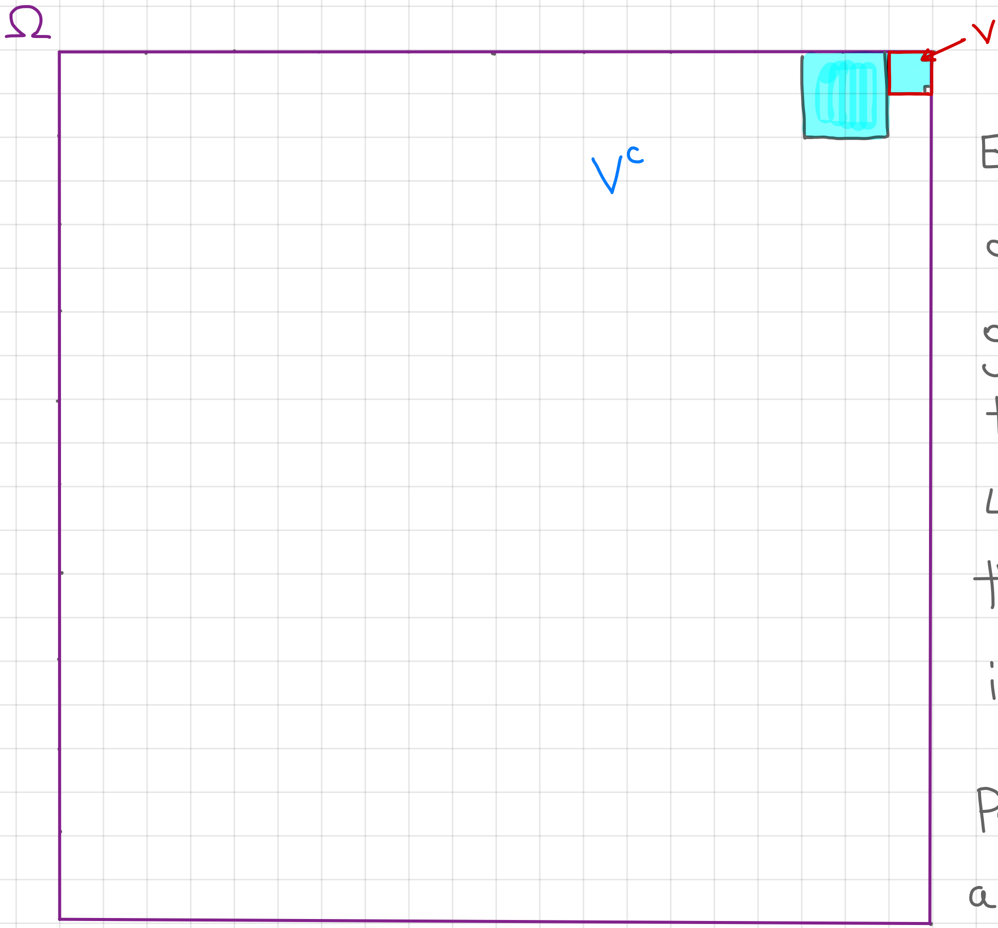
(e) not enough information

$T = \{\text{positive test}\}$ $\quad P(T \mid V^c) = 0.01 \quad P(T^c \mid V) = 0.01$

$V = \{\text{has virus}\}$ $\quad P(V) = 0.0025$

$\Omega = V \cup V^c$ $\quad P(T \mid V) = 1 - P(T^c \mid V) = 0.99$

$$P(V \mid T) = \frac{P(T \mid V)\,P(V)}{P(T)} = \frac{P(T \mid V)\,P(V)}{P(T \mid V)\,P(V) + P(T \mid V^c)\,P(V^c)} = \frac{0.99 \cdot 0.0025}{0.99 \cdot 0.0025 + 0.01 \cdot (0.9975)} = 0.1987 \approx 20\%$$



: test positive

Even though only 1% of individuals in V^c get (false) positive test results, it is still 4 times more people than 99% of individuals in V that test positive.

Posterior probabilities are highly sensitive to prior inputs!

What if $P(V) = 2.5\%$? $P(V|T) = 72\%$

$P(V) = 25\%$? $P(V|T) = 97\%$