MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Conditional probability. Independence

Next: ASV 3.1

Week 3:

- homework 1 (due Monday, January 23)

Conditional probability

Def. Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$ the conditional probability of A given B is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

If $P(B) > 0$, then the conditional probability $P(\cdot \mid B)$ satisfies all the properties of probability:
axioms of probability, probability of the complement, monotonicity, inclusion-exclusion ...

Remark If Ω is finite and P is uniform, then

$$P(A \mid B) = \frac{\# A \cap B}{\# B}$$

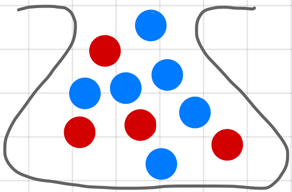
Multiplication rule

By definition $P(B|A) = \dfrac{P(A\cap B)}{P(A)}$

$$\Rightarrow P(A\cap B) = P(A)\cdot P(B|A) \leftarrow \text{multiplication rule}$$

For $A\cap B\cap C$: $P(A\cap B\cap C) = P(AB)\,P(C|AB) = P(A)P(A|B)P(C|AB)$

Example

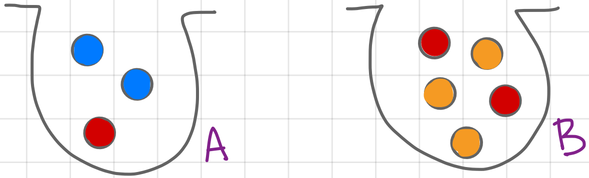


An urn contains 4 red ball and 6 blue balls. Two balls are sampled without replacement. What is the probability that both are red?

Two-stage experiments

- perform an experiment, measure a random outcome
- perform a second experiment whose setup depends on the outcome of the first!

Example



- first choose an urn at random
- then sample a ball at random from the chosen urn

Q: What is the probability that the sampled ball is red?

$R = \{\text{sample red ball}\}$, $A = \{\text{choose urn } A\}$, $B = \{\text{choose urn } B\}$

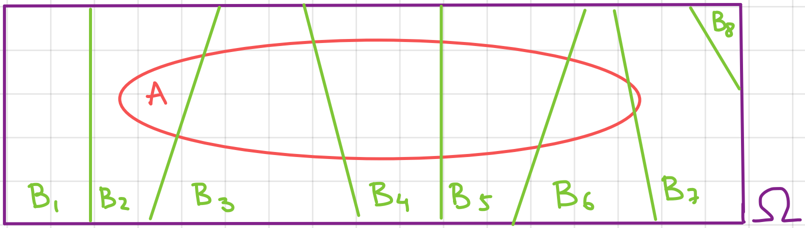
Law of total probability

Let $B_1, B_2, \ldots, B_n$ be a partition of Ω

(i.e., B_i are disjoint, $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$, $P(B_i) > 0$).

Then for every event A:

$$P(A) = P(AB_1 \cup AB_2 \cup \cdots \cup AB_n)$$



<u>Example</u> 90% of coins are fair, 9% are biased to come up heads 60% of times, 1% are biased to come up heads 80%. You find a coin on the street.

How likely is it to come up heads?

Law of total probability

Define $A = \{\text{coin comes up heads}\}$, $B_1 = \{\text{coin is fair}\}$

$B_2 = \{\text{coin is 60\% biased}\}$ $B_3 = \{\text{coin is 80\% biased}\}$

- B_1, B_2, B_3 form a and

- $P(A \mid B_1) =$ $P(A \mid B_2) =$ $P(A \mid B_3) =$

Then using the law of total probability

$$P(A) =$$

$$=$$

Another question: In the same setting, you find a coin and toss it. It comes up heads. How likely is it that this coin is 80% biased (heavily biased)?

Important remark

We know that $P(A|B_3) = 0.8$.

What can we say about $P(B_3|A)$?

Generally speaking,

Example According to Forbes, there are 2668 billionaires in the world, 2357 of them are men.

Example Prosecutor's fallacy:

$E = \{\text{evidence on the defendant}\}$

$I = \{\text{defendant is innocent}\}$

Bayes' Rule (relation between $P(A|B)$ and $P(B|A)$)

$$P(B|A) =$$

This formula is often used with the law of total probability.

Let $B_1, B_2, \ldots, B_n$ be a partition of the sample space. Then for any event A with $P(A) > 0$

$$P(B_k|A) =$$

Example. $A = \{\text{coin comes up heads}\}$, $B_1 = \{\text{coin is fair}\}$

$B_2 = \{\text{coin is 60\% biased}\}$ $B_3 = \{\text{coin is 80\% biased}\}$

Given: $P(B_1) =$, $P(B_2) =$, $P(B_3) =$ $P(A|B_1) =$, $P(A|B_2) =$,

$P(A|B_3) =$

We have computed that $P(A) = \sum_{i=1}^{3} P(B_i) P(A|B_i) = 0.512$

Then

Bayes' rule

Example

Suppose that a certain test (e.g., virus X test) is 99% accurate (1% false positives, 1% false negatives). 0.25% of the population have this virus.

You test positive. What is the probability you have this virus?

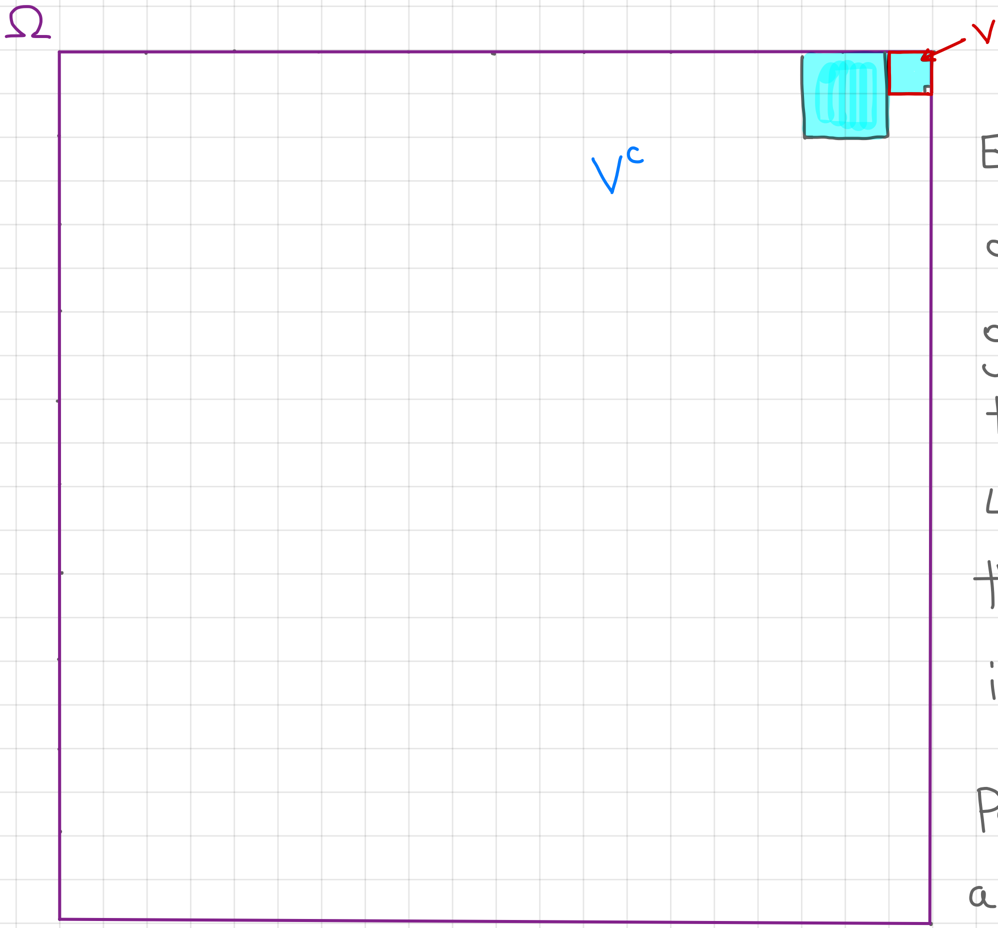
(a) 99%

(b) 20%

(c) 1%

(d) 0.3%

(e) not enough information



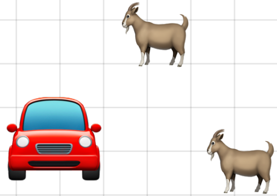
: test positive

Even though only 1% of individuals in V^c get (false) positive test results, it is still 4 times more people than 99% of individuals in V that test positive.

Posterior probabilities are highly sensitive to prior inputs!

What if

The Monty Hall Problem



You play the following game. There are three doors. Each door hides a prize. Behind one door there is a car, behind two other doors - goats.

(not open)

You choose one door. The host opens one of the doors you did not choose, revealing a goat. You are now given a possibility to either stick with your original choice, or switch to the other closed door.

Should you switch? (a) Yes (b) No (c) Doesn't matter

The Monty Hall Problem

Let's call the door you choose #1. In this case Monty will open door #2 or door #3. Suppose Monty opens #2.

$B_i = \{\text{the car is behind door } \# i\}$

$A = \{\text{Monty opens door } \# 2\}$

We want to know

$P(B_3 | A) =$

Independence

We have seen how knowing that event B occurred may change the probability of event A, $P(A)$ vs. $P(A|B)$

What if we have two events A and B that have nothing to do with each other?

! A and B have nothing to do with each other is not the same as A and B being disjoint!

Example Flip a coin 3 times. $A = \{$the first coin is heads$\}$
$B = \{$the second coin is tails$\}$

$A = \{HHH, HHT, HTH, HTT\}$ The first toss has no influence

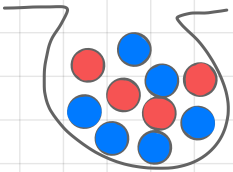
$B = \{TTT, TTH, HTH, HTT\}$ on the second toss

Independence

Def Two events A and B are (statistically) independent

if

Example



An urn has 4 red and 6 blue balls.

Two balls are sampled.

$A = \{ 1^{st} \text{ ball is red} \}$

$B = \{ 2^{nd} \text{ ball is blue} \}$

Are A and B independent?

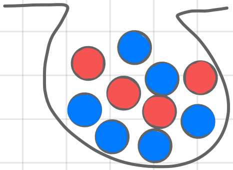
(a) Yes

(b) No

(c) Not enough information

Independence

Example



An urn has 4 red and 6 blue balls.

Two balls are sampled.

$A = \{1^{st} \text{ ball is red}\}$ $B = \{2^{nd} \text{ ball is blue}\}$

Are A and B independent?

1) choose balls with replacement

$P(A) =$

$P(A \cap B) =$

$P(B) =$

2) choose balls without replacement

$P(A) =$

$P(A \cap B) =$

$P(B) =$

A and B are