MATH 180A (Lecture A00)

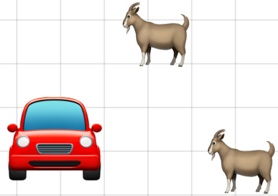
mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Independence.
Random variables
Next: ASV 3.2

Week 3:

- homework 2 (due Sunday, January 29)
- Midterm 1 (Wednesday, February 1, lectures 1-8)
- 5 homework extension days per student per quarter

The Monty Hall Problem



You play the following game. There are three doors. Each door hides a prize. Behind one door there is a car, behind two other doors – goats.

(not open)

You choose one door. The host opens one of the doors you did not choose, revealing a goat. You are now given a possibility to either stick with your original choice, or switch to the other closed door.

Should you switch? (a) Yes (b) No (c) Doesn't matter

The Monty Hall Problem

Let's call the door you choose #1. In this case Monty will open door #2 or door #3. Suppose Monty opens #2.

$B_i = \{$ the car is behind door #$i\}$ $\quad P(B_i) = \frac{1}{3}$

$A = \{$ Monty opens door #2$\}$ $\quad P(A \mid B_2) = 0$

We want to know $P(B_3 \mid A)$. $\quad P(A \mid B_3) = 1$

$P(A \mid B_1) = \frac{1}{2}$

$$P(B_3 \mid A) = \frac{P(B_3 \cap A)}{P(A)} = \frac{P(B_3)P(A \mid B_3)}{\sum_{i=1}^{3} P(B_i)\, P(A \mid B_i)} = \frac{\frac{1}{3}\cdot 1}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1} = \frac{1}{\frac{3}{2}} = \frac{2}{3} > \frac{1}{2}$$

Independence

We have seen how knowing that event B occurred may change the probability of event A, $P(A)$ vs. $P(A|B)$

What if we have two events A and B that have nothing to do with each other?

! A and B have nothing to do with each other is not the same as A and B being disjoint!

Example Flip a coin 3 times. A = {the first coin is heads}
B = {the second coin is tails}

$A = \{HHH, HHT, HTH, HTT\}$ The first toss has no influence

$B = \{TTT, TTH, HTH, HTT\}$ on the second toss

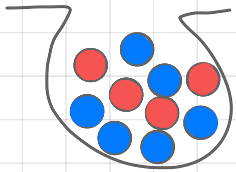
$P(A \cap B) = \frac{2}{8} = \frac{1}{4}$, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = P(B)$

Independence

Def Two events A and B are (statistically) independent if

$$P(A \cap B) = P(A)P(B)$$

Example



An urn has 4 red and 6 blue balls.

Two balls are sampled.

$A = \{ 1^{st} \text{ ball is red} \}$

$B = \{ 2^{nd} \text{ ball is blue} \}$

Are A and B independent?

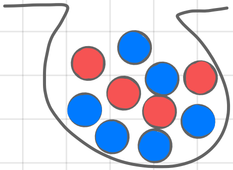
(a) Yes

(b) No

(c) Not enough information

Independence

Example



An urn has 4 red and 6 blue balls.

Two balls are sampled.

$A = \{1^{st} \text{ ball is red}\}$ $B = \{2^{nd} \text{ ball is blue}\}$

Are A and B independent?

1) choose balls with replacement

$$P(A) = \frac{4 \cdot 10}{10 \cdot 10} = 0.4$$

$$P(A \cap B) = \frac{4 \cdot 6}{10 \cdot 10} = 0.24 = P(A)P(B)$$

$$P(B) = \frac{10 \cdot 6}{10 \cdot 10} = 0.6$$

2) choose balls without replacement

$$P(A) = \frac{4 \cdot 9}{10 \cdot 9} = \frac{4}{10}$$

$$P(A \cap B) = \frac{24}{10 \cdot 9} \neq \frac{4}{10} \cdot \frac{6}{10} = P(A)P(B)$$

$$P(B) = \frac{9 \cdot 6}{10 \cdot 9} = \frac{6}{10}$$

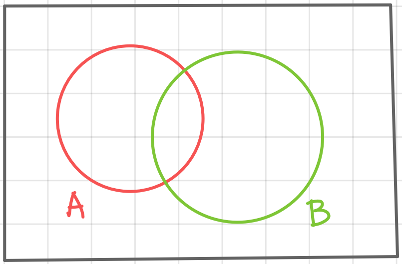
A and B are not independent

Independence

$P(A) = P(A \cap B) + P(A \cap B^c)$

A and B are independent if and only if A and B^c are independent

Proof. ($\Rightarrow$) Suppose that A and B are independent



$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$\overset{\text{indep}}{=} P(A) - P(A)P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A)P(B^c) \Rightarrow A \text{ and } B^c \text{ are independent}$$

($\Leftarrow$)

Independence for more than two events

Def. A collection of events $A_1, A_2, \dots, A_n$ is **mutually independent** if for any subcollection of events $A_{i_1}, A_{i_2}, \dots, A_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

Example For $n=3$, A, B, C are mutually independent if

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Suppose that A and B are independent, A and C are independent, B and C are independent. Are A, B and C mutually independent?

Important example

Toss a coin

A = { there is exactly one tails in the first two tosses }

B = { there is exactly one tails in the last two tosses }

C = { there is exactly one tails in the first and last tosses }

$A = \{(H,T,*),(T,H,*)\}$ $B = \{(*,H,T),(*,T,H)\}$

$C = \{(H,*,T),(T,*,H)\}$

$P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C)$

$P(A \cap B) =$ $P(B \cap C)$ $P(A \cap C)$,

$P(A \cap B \cap C) =$