Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 8 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (10 points) You have an urn that initially contains 6 red balls, 2 black balls and 1 green ball. On the first step, you choose one ball uniformly at random from the urn, look at its color, and then return it back to the urn together with one more ball of the same color (e.g., if you pick a red ball, then you put it back to the urn together with another red ball). Then on the second step you choose a ball uniformly at random from the urn (note that on the second step the urn contains the additional ball).

   What is the probability that on the second step you choose a red ball?

   Solution.

   Denote

$$A = \{\text{ball chosen on the first step is red}\}, \quad B = \{\text{ball chosen on the second step is red}\}. \tag{1}$$

   Then using the law of total probability

$$P(B) = P(B|A)P(A) + P(B|A^C)P(A^C) = \frac{7}{10} \cdot \frac{6}{9} + \frac{6}{10} \cdot \frac{3}{9} = \frac{2}{3}. \tag{2}$$

2. (10 points) Every morning Frank chooses how to commute to work: by car or by bicycle. He chooses bicycle with probability 0.7. The probability that Frank is late to work if he rides a bicycle is 0.1, and the probability that he is late if he drives a car is 0.2. Frank is late today. What is the probability, that he came to work by bicycle?

Solution. Denote events:

$$B = \{\text{Frank goes by bicycle}\}, \tag{3}$$
$$C = B^C = \{\text{Frank goes by car}\}, \tag{4}$$
$$L = \{\text{Frank is late}\}. \tag{5}$$

It is given that

$$P(B) = 0.7, \quad P(L|B) = 0.1, \quad P(L|B^C) = 0.2. \tag{6}$$

We have to find $P(B|L)$.

From Bayes' formula we have

$$P(B|L) = \frac{P(L|B)P(B)}{P(L|B)P(B) + P(L|B^C)P(B^C)} \tag{7}$$
$$= \frac{0.1 \cdot 0.7}{0.1 \cdot 0.7 + 0.2 \cdot 0.3} = \frac{0.07}{0.13} = \frac{7}{13}. \tag{8}$$

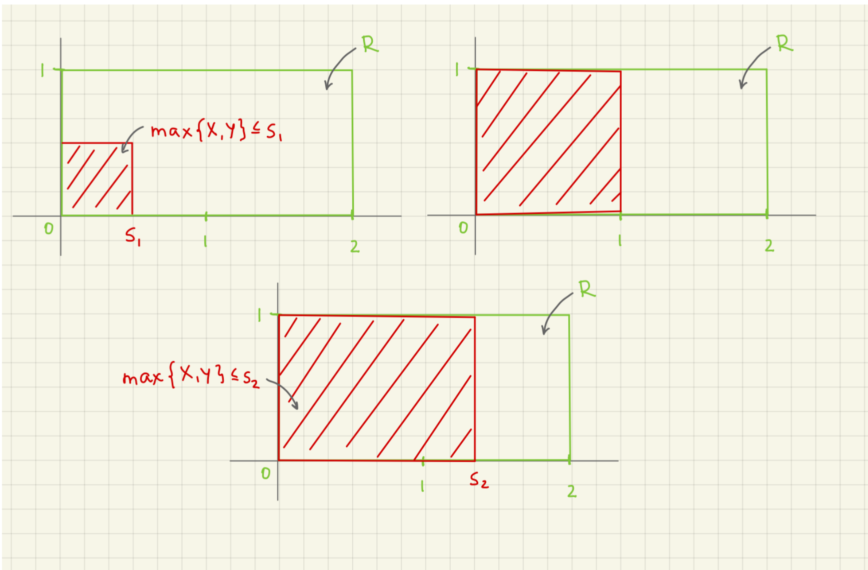
3. (10 points) Consider a point $P = (X, Y)$ chosen uniformly at random inside the rectangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$ and $(2, 1)$. Let $Z = \max(X, Y)$ be the random variable defined as the maximum of the two coordinates of the point. [**Hint**. Draw a picture.]

   (a) Compute and plot the cumulative distribution function of Z.

   Solution. Denote $Z = \max\{X, Y\}$ and $R = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2, 0 \le y \le 1\}$. Denote also $A_s = \{(x, y) \in \mathbb{R}^2 : \max\{x, y\} \le s\}$. Then from the definition of the uniform distribution

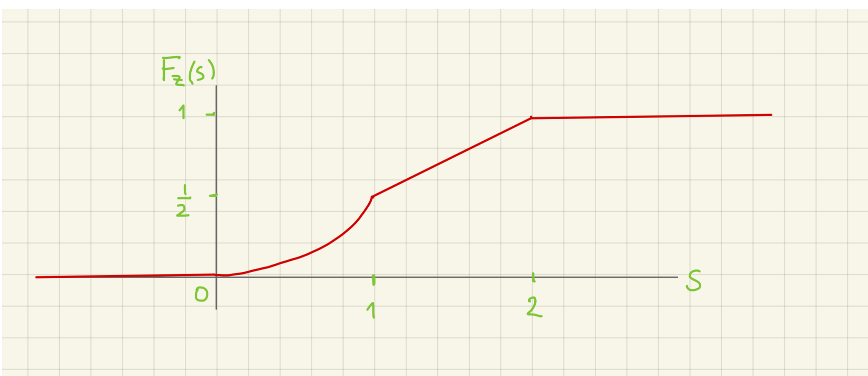
$$P(Z \le s) = P((X, Y) \in A_s) = \frac{\text{Area}(A_s)}{\text{Area}(R)}. \tag{9}$$

   For $0 \le s \le 1$ we have $\text{Area}(A_s) = s^2$, and for $1 \le s \le 2$ we have $\text{Area}(A_s) = s$ (see the picture below)



   Then

$$F_Z(s) = P(Z \le s) = \begin{cases} 0, & s < 0, \\ \frac{s^2}{2}, & 0 \le s < 1, \\ \frac{s}{2}, & 1 \le s < 2, \\ 1, & s \ge 2. \end{cases} \tag{10}$$



(b) Determine if Z is continuous, discrete or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

Solution. Random variable Z is continuous and its density is

$$f_Z(s) = \begin{cases} 0, & s < 0, \\ s, & 0 \le s < 1, \\ \frac{1}{2}, & 1 \le s < 2, \\ 0, & s \ge 2. \end{cases} \tag{11}$$

4. (10 points) A study showed that 2% of San Diego residents own a boat.
   (a) Estimate the probability that among 100 randomly interviewed San Diego residents there are at least 3 boat owners.
   (b) Explain why the approximation that you used in part (a) is better compared to other approximations that you know.

[For full credit, present your answer in the closed form (not as an infinite series); you may leave your answer in terms of e^x or $\Phi(x)$]

Solution.

(a) If X is the number is interviewed residents of San Diego that own boat. Then $X \sim \text{Bin}(100, 0.02)$. To compute the probability $P(X \geq 3)$ we first rewrite it as

$$P(X \geq 3) = 1 - P(X \leq 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2), \tag{12}$$

and then use the Poisson approximation of the binomial distribution with $\lambda = 100{\cdot}0.02 = 2$

$$P(X \geq 2) \approx 1 - e^{-2} - 2e^{-2} - \frac{2^2}{2}e^{-2} = 1 - 5e^{-2}. \tag{13}$$

(b) Using the criterion from Lecture 14, compute the bounds for the approximation error for normal and Poisson approximations of $\text{Bin}(100, 0.02)$. The error in the Poisson approximation is bounded by

$$np^2 = 100 \cdot (0.02)^2 = 0.04, \tag{14}$$

while the error in the normal approximation is bounded by

$$\frac{1}{\sqrt{np(1-p)}} = \frac{1}{\sqrt{100 \cdot 0.02 \cdot 0.98}} \approx 0.7. \tag{15}$$

By comparing the above numbers it is clear that the Poisson distribution ensures better approximation.

5. (10 points) Let X and Y be independent random variables uniformly distributed on the interval $[0,1]$, i.e., $X \sim \mathcal{U}[0,1]$, $Y \sim \mathcal{U}[0,1]$.
   (a) Compute the moment generating function of the sum $X+Y$.
   (b) Show that for any $t \in \mathbb{R}$

$$(e^t-1)^2 = e^{2t} - 2e^t + 1 = \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k - \sum_{k=2}^{\infty} \frac{2}{k!} t^k. \tag{16}$$

   (c) Use the results of (a) and (b) to compute $E((X+Y)^n)$, moments of the sum, for any $n \in \mathbb{N}$.

Solution.

(a) Compute the moment generating function: for any $t \in \mathbb{R}$

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY}) = \int_0^1 e^{tx}dx \int_0^1 e^{ty}dy = \frac{(e^t-1)^2}{t^2} \tag{17}$$

(b) The first equality is the expansion of the square of a difference, and the second equality follows from the power series expansion of the exponential function

$$e^{2t} = 1 + 2t + \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k, \quad 2e^t = 2 + 2t + \sum_{k=2}^{\infty} \frac{2}{k!} t^k, \tag{18}$$

$$e^{2t} - 2e^t + 1 = (1 + 2t - 2 - 2t + 1) + \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k - \sum_{k=2}^{\infty} \frac{2}{k!} t^k. \tag{19}$$

(c) From (a) and (b) we have that

$$M_{X+Y}(t) = \frac{\sum_{k=2}^{\infty} \frac{2^k-2}{k!} t^k}{t^2} = \sum_{k=2}^{\infty} \frac{2^k-2}{k!} t^{k-2}, \tag{20}$$

by changing $n = k-2$

$$\sum_{k=2}^{\infty} \frac{2^k-2}{k!} t^{k-2} = \sum_{n=0}^{\infty} \frac{2^{n+2}-2}{(n+1)(n+2)} \cdot \frac{t^n}{n!} \tag{21}$$

and we conclude that

$$E((X+Y)^n) = \frac{2^{n+2}-2}{(n+1)(n+2)}. \tag{22}$$

6. (10 points) Suppose that $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(q)$ are independent random variables. Find the probability $P(X < Y)$.

Solution.

We decompose the desired probability as

$$P(X < Y) = \sum_{k=1}^{\infty} P(X < Y, X = k) = \sum_{k=1}^{\infty} P(X = k, Y > k) = \sum_{k=1}^{\infty} P(X = k)P(Y > k).$$

Since $X \sim \text{Geom}(p)$, we have $P(X = k) = p(1-p)^{k-1}$. Similarly, since $Y \sim \text{Geom}(q)$, we have $P(Y > k) = (1-q)^k$. So,

$$\begin{aligned} P(X < Y) &= \sum_{k=1}^{\infty} P(X = k)P(Y > k) \\ &= \sum_{k=1}^{\infty} p(1-p)^{k-1}(1-q)^k \\ &= p(1-q)\sum_{k=1}^{\infty}((1-p)(1-q))^{k-1} \\ &= p(1-q)\frac{1}{1-(1-p)(1-q)}. \end{aligned}$$

7. (10 points) Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with $X_1 \sim \text{Unif}[0,1]$. Let $Y = \min(X_1, \dots, X_n)$. Find the CDF F_Y and density f_Y.

Solution.

Note that $Y \in [0,1]$. So, $F_Y(t) = 0$ if $t < 0$ and $F_Y(t) = 1$ if $t > 1$. For $t \in [0,1]$,

$$\begin{aligned} F_Y(t) &= P(Y \leq t) \\ &= 1 - P(Y > t) \\ &= 1 - P(\min(X_1, \dots, X_n) > t) \\ &= 1 - \prod_{i=1}^{n} P(X_i > t) = 1 - (1-t)^n. \end{aligned}$$

Differentiating, we see that

$$f_Y(t) = \begin{cases} n(1-t)^{n-1} & \text{if } t \in [0,1]; \\ 0 & \text{if } t \notin [0,1]. \end{cases}$$

8. (10 points) Let T be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, and $(1,1)$ (including the interior). Suppose that $P = (X, Y)$ is a point chosen uniformly at random inside of T.

   (a) What is the joint density function of (X, Y)? Use this to compute $\text{Cov}(X, Y)$.

   Solution.
   The joint density of (X, Y) is

$$f_{(X,Y)} = \begin{cases} \frac{1}{\text{Area}(T)} = 2 & \text{if } (x,y) \in T; \\ 0 & \text{if } (x,y) \notin T. \end{cases}$$

   We compute the covariance using the formula $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$:

$$\begin{aligned} E(XY) &= \int_0^1 \int_x^1 2xy \, dydx \\ &= \int_0^1 \left(xy^2\right)\Big|_x^1 dy \\ &= \int_0^1 x - x^3 \, dy \\ &= \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

   Similarly,

$$\begin{aligned} E(X) &= \int_0^1 \int_x^1 2x \, dydx \\ &= \int_0^1 2x - 2x^2 \, dx \\ &= \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

   and

$$\begin{aligned} E(Y) &= \int_0^1 \int_x^1 2y \, dydx \\ &= \int_0^1 \left(y^2\Big|_x^1\right) dx \\ &= \int_0^1 1 - x^2 \, dx \\ &= \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

   So, $\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{3}\frac{2}{3} = \frac{1}{4} - \frac{2}{9} > 0$.

   (b) Determine if X and Y are independent.

   Solution.
   X and Y are not independent because $\text{Cov}(X,Y) \neq 0$.