MATH 180A (Lecture A00 - NEMISH) MIDTERM 2 03/01/23, 12-12:50pm, MANDE B-210

Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

1. (20 points) Let X be a continuous random variable with the probability density function

$$f_X(x) = \begin{cases} c \cdot (2+x)^2, & x \in (0,1), \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 0$ is a constant that will be determined later. Let $Y = \frac{1}{2+X}$.

[You may leave the answers to parts (a) and (b) without writing explicitly the value of c.]

(a) Compute $E(Y)$.

Solution. Using the formula for the expectation of a function of a random variable, we get

$$E(Y) = E\Big(\frac{1}{2+X}\Big) = \int_{-\infty}^{\infty} \frac{1}{2+x} f_X(x)dx = \int_0^1 \frac{1}{2+x} c(2+x)^2 dx = c\int_0^1 (2+x)dx = \frac{5c}{2}.$$

(b) Compute $\mathrm{Var}(Y)$.

Solution. We use the formula $\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$. First compute $E(Y^2)$

$$E(Y^2) = E\Big(\frac{1}{(2+X)^2}\Big) = \int_{-\infty}^{+\infty} \frac{1}{(2+x)^2} f_X(x)dx = \int_0^1 \frac{1}{(2+x)^2} c(2+x)^2 dx = \int_0^1 c dx = c,$$

which together with part (a) gives

$$\mathrm{Var}(Y) = c - \frac{25}{4}c^2 = c\Big(1 - \frac{25c}{4}\Big). \tag{1}$$

(c) Determine the value of c.

Solution. Finally, from the properties of the density function $\int_{\mathbb{R}} f_X(x)dx = 1$ we get that

$$\int_0^1 c(2+x)^2 dx = c\frac{(2+x)^3}{3}\Big|_0^1 = c \cdot \frac{19}{3} = 1 \quad \Rightarrow \quad c = \frac{3}{19}. \tag{2}$$

2. (20 points) A research company conducted a phone poll in the city of San Diego to study the food preferences of the city residents. The interviewees were asked to choose which meal they prefer: burritos or tacos. Out of 1600 interviewed residents, 1200 replied that they prefer tacos.

   Using the above data, construct a 95% confidence interval for the (unknown) fraction of the population that prefers tacos. Clearly state (in words) your conclusion.

   [Leave your answer in terms of Φ.]

   Solution. Let p denote the fraction of the population that prefers tacos. Then we use the normal approximation of the binomial distribution to construct the confidence interval

$$P(|p-\widehat{p}|<\varepsilon) \geq 0.95, \tag{3}$$

   where $\widehat{p} = 1200/1600 = 0.75$. It has been shown in Lecture 18 that

$$P(|p-\widehat{p}|<\varepsilon) \geq 2\Phi(2\varepsilon\sqrt{n}) - 1, \tag{4}$$

   where n is the number of experiments (number of interviewed San Diegans). In order to find the smallest interval (the smallest $\varepsilon > 0$) that guarantees (3), we take ε such that

$$2\Phi(2\varepsilon\sqrt{n}) - 1 = 0.95. \tag{5}$$

   Solving (5) with respect to ε gives

$$\varepsilon = \frac{\Phi^{-1}(0.975)}{80}. \tag{6}$$

   We conclude that

$$\left(0.75 - \frac{\Phi^{-1}(0.975)}{80}, 0.75 + \frac{\Phi^{-1}(0.975)}{80}\right) \tag{7}$$

   gives the 95%-confidence interval for the unknown fraction of San Diegans who prefer tacos.

3. (20 points) The final exam consists of 162 multiple choice questions. Each question has three answer options, only one of which is correct. In order to pass the exam, students have to answer correctly at least 75 questions.

   A student decides not to study at all for the exam and choose the answers at random. Estimate the probability that this student passes the exam.

   [Explain your choice of approximation. You may leave the answers in terms of Φ. Do not use the continuity correction.]

   Solution. Denote by X the number of correct answers given by student A. The probability of successfully answering each question is $1/3$, and the outcomes for different questions are independent. Therefore, X has binomial distribution with parameters $n = 162$ and $p = 1/3$, $X \sim \text{Bin}(162, 1/3)$. Then we have to estimate the probability

$$P(\text{student passes the exam}) = P(X \geq 75). \tag{8}$$

   We check if the normal approximation is applicable in this situation by computing $np(1-p)$

$$np(1-p) = 162 \cdot \frac{1}{3} \cdot \frac{2}{3} = 36 > 10. \tag{9}$$

   Using the rule of thumb we conclude that the normal approximation should give a good approximation of the probability (8).

   Using the normal approximation (without applying the continuity correction), we get

$$P(X \geq 75) = P\Big(\frac{X - 162/3}{\sqrt{36}} \geq \frac{75-54}{6}\Big) \approx P\Big(Z \geq \frac{7}{2}\Big) = 1 - \Phi\Big(\frac{7}{2}\Big), \tag{10}$$

   where $Z \sim \mathcal{N}(0,1)$.

   We conclude that the student passes the exam with probability around $1 - \Phi(\frac{7}{2}) \approx 0.0002$.

4. (20 points) Felix has one dollar and a biased coin. When the coin is tossed, heads comes up with probability $p > 1/2$.

   Felix plays the following game. He tosses the coin. If heads comes up, he stops. If tails comes up, he continues tossing until heads comes up. After each toss his fortune (money he possesses before tossing the coin) doubles. For example, if the first toss is tails and the second toss is heads, Felix ends the game after the two tosses with 4 dollars.

   (a) Compute Felix's expected fortune at the end of the game.

   Solution. Let X denote the number of the tosses it takes Felix to see heads for the first time. Then X has geometric distribution with parameter p. Since at the beginning of the game Felix has one dollar, and his fortune doubles after each toss, at the end of the game Felix has 2^X dollars. Using the formula for the expectation of a function of a discrete random variable from Lecture 14, we compute Felix's expected fortune at the end of the game

$$E(2^X) = \sum_{k=1}^{\infty} 2^k (1-p)^{k-1} p = 2p \sum_{\ell=0}^{\infty} (2(1-p))^{\ell} = \frac{2p}{1-2(1-p)} = \frac{2p}{2p-1}. \qquad (11)$$

   (b) Compute the probability that Felix wins more than 20 dollars. [For full credit, present your answer in the closed form (not as an infinite series)]

   Solution. Felix wins more than 20 dollars if and only if he tosses the coin at least 5 times, so

$$P(\text{wins more than 20 dollars}) = P(X \geq 5) = \sum_{k=5}^{\infty} p(1-p)^{k-1} = (1-p)^4. \qquad (12)$$