# MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

## Today: Conditioning on continuous random variables

## Next: PK 7.1, Durrett 3.1

Week 4:

- HW4 due Friday, May 5 on Gradescope

# Properties of conditional probability / expectation

1) $$P(a < X < b, c < Y < d) = \int_c^d \left( \int_a^b f_{X|Y}(x|y)\,dx \right) f_Y(y)\,dy$$

$$= \int_c^d P(X \in (a,b) \mid Y = y)\, f_Y(y)\,dy$$

2) $$P(a < X < b) = \int_{-\infty}^{+\infty} \left( \int_a^b f_{X|Y}(x|y)\,dx\, f_Y(y)\,dy \right.$$

$$= \int_{-\infty}^{+\infty} P(X \in (a,b) \mid Y = y)\, f_Y(y)\,dy$$

3) $$E(g(X)) = \int_{-\infty}^{+\infty} E(g(X) \mid Y = y)\, f_Y(y)\,dy$$

## Further properties of conditional expectation (PK, p.50)

4) $E(c_1 g_1(X_1) + c_2 g_2(X_2) \mid Y=y) = c_1 E(g_1(X_1) \mid Y=y) + c_2 E(g_2(X_2) \mid Y=y)$

5) $E(v(X,Y) \mid Y=y) = E(v(X,y) \mid Y=y)$

In particular, $E(v(X,Y)) = \int_{-\infty}^{+\infty} E(v(X,y) \mid Y=y) f_Y(y)\,dy$

6) $E(g(X)h(Y)) = \int_{-\infty}^{+\infty} h(y) E(g(X) \mid Y=y) f_Y(y)\,dy$

$= E(h(Y) E(g(X) \mid Y))$

7) $E(g(X) \mid Y=y) = E(g(X))$ if $X$ and $Y$ are independent

## Example 1

Let $(X, Y)$ be jointly continuous r.v.s with density

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y} - y}, \quad x, y > 0$$

Compute the conditional density of $X$ given $Y=y$.

1) Compute the marginal density of $Y$

$$f_Y(y) = \int_0^{+\infty} \frac{1}{y} e^{-\frac{x}{y} - y} \, dx = e^{-y} \underbrace{\int_0^{\infty} \frac{1}{y} e^{-\frac{x}{y}} \, dx}_{1} = e^{-y}, \quad (Y \sim Exp(1))$$

2) Compute the conditional density given $Y=y$

$$f_{X|Y}(x|y) = \frac{\frac{1}{y} e^{-\frac{x}{y} - y}}{e^{-y}} = \frac{1}{y} e^{-\frac{x}{y}}$$

$X \sim Exp\left(\frac{1}{y}\right)$

## Example 1 (cont.)

Suppose that $Y \sim \mathrm{Exp}(1)$, and $X$ has exponential distribution with parameter $\frac{1}{Y}$. Compute $E(X)$

First, $E(X \mid Y=y) = y$, and using protery 3)

$$E(X) \overset{(3)}{=} \int_0^\infty E(X \mid Y=y)\, f_Y(y)\, dy$$

$$= \int_0^\infty y\, e^{-y}\, dy = 1$$

$$E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-\frac{x}{y} - y}\, dx\, dy$$

## Example 2: continuous and discrete r.v.s

Let $N \in \mathbb{N}$, $P \sim \mathcal{U}nif[0,1]$, $X \sim Bin(N, P)$

What is the distribution of $X$?

$$P(X=k) = \int_0^1 P(X=k \mid P=s)\, f_P(s)\, ds$$

$$= \int_0^1 P(X=k \mid P=s) \cdot 1\, ds$$

$$= \int_0^1 \binom{N}{k} s^k (1-s)^{N-k}\, ds$$

$$= \frac{N!}{k!\,(N-k)!} \cdot \frac{k!\,(N-k)!}{(N+1)!} = \frac{1}{N+1}$$

$\Rightarrow$ $X$ is uniformly distributed on $\{0, \dots, N\}$

## Example 3

Let $X$ and $Y$ be i.i.d. $Exp(\lambda)$ r.v.

Define $Z = \frac{X}{Y}$. Compute the density of $Z$.

- If $X \sim Exp(\lambda)$, then for $\alpha > 0$ $\alpha X \sim Exp\left(\frac{\lambda}{\alpha}\right)$

$$P(\alpha X > t) = P\left(X > \frac{t}{\alpha}\right) = e^{-\lambda \frac{t}{\alpha}} = e^{-\frac{\lambda}{\alpha} t} \Rightarrow \alpha X \sim Exp\left(\frac{\lambda}{\alpha}\right)$$

- $$P(Z > t) = \int_0^{+\infty} P(Z > t \mid Y = y) f_Y(y)\, dy$$

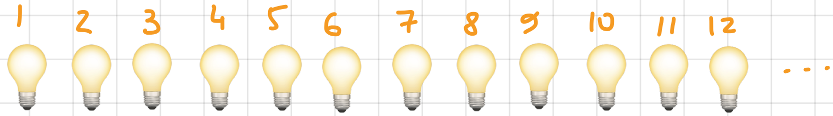
$$= \int_0^{+\infty} P\left(\frac{X}{Y} > t \mid Y = y\right) \lambda e^{-\lambda y}\, dy$$

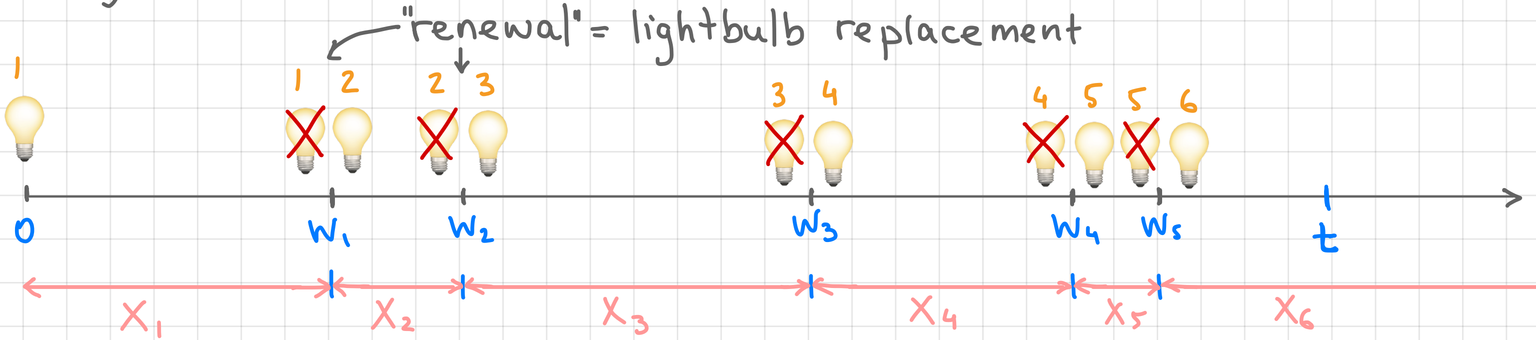
$$= \int_0^{\infty} P\left(\frac{1}{y} X > t\right) \lambda e^{-\lambda y}\, dy = \int_0^{+\infty} e^{-\lambda y t} \lambda e^{-\lambda y}\, dy$$

$$= \lambda \int_0^{+\infty} e^{-\lambda(1+t) y}\, dy = \frac{1}{1+t} \quad \Rightarrow f_Z(t) = \frac{1}{(1+t)^2}$$

# Renewal process

Imagine lightbulbs




$X_i$ – lifetime of the lightbulb #i, $W_i$ = time of i-th "renewal"

Lightbulbs are identical $\Rightarrow$ $X_i$ are i.i.d.

Let $N(t)$ denote the number of renewals up to time $t$

- What are the properties of $(N(t))_{t\geq 0}$?
- How they depend on the distribution of $X_i$?

# Renewal process. Definition

**Def.** Let $\{X_i\}_{i\geq 1}$ be i.i.d. r.v.s, $X_i > 0$.

Denote $W_n := X_1 + \cdots + X_n$, $n \geq 1$, and $W_0 := 0$.

We call the counting process

$$N(t) = \#\{k > 0 : W_k \leq t\} = \max\{n : W_n \leq t\}$$

the renewal process.

**Remarks.** 1) $W_n$ are called the waiting/renewall times

$X_i$ are called the interrenewal times

2) $N(t)$ is characterised by the distribution of $X_i > 0$

3) More generally, we can define for $0 \leq a < b < \infty$

$$N((a,b]) = \#\{k : a < W_k \leq b\}$$