MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Poisson process as a renewal process

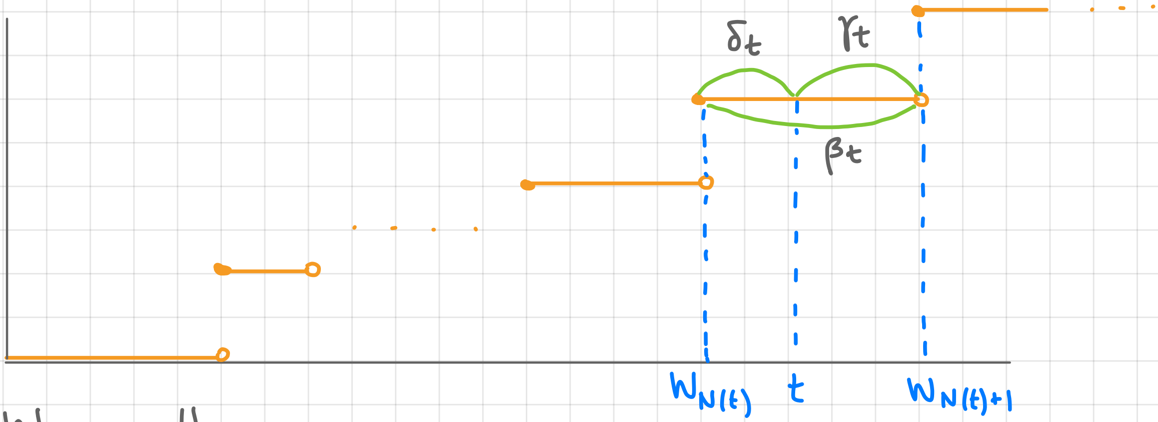
Next: PK 7.2-7.3, Durrett 3.1

Week 6:

- HW5 due Friday, May 12 on Gradescope

Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

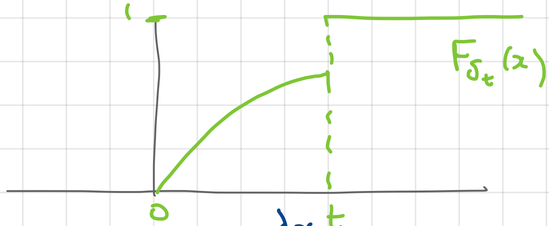
Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life $\quad \gamma_t \sim \text{Exp}(\lambda)$

$$P(\gamma_t > x) = P\big(N(t+x) - N(t) = 0\big) = P\big(N(x) = 0\big) = e^{-\lambda x}$$

- current life $\quad \delta_t$



$$P(\delta_t > x) = \begin{cases} 0, & \text{if } x \geq t \\ P\big(N(t-x) = N(t)\big) = P\big(N(x) = 0\big) = e^{-\lambda x} \end{cases}$$

- total life $\quad \beta_t = \gamma_t + \delta_t$

$$E(\gamma_t + \delta_t) = E(\gamma_t) + E(\delta_t) = \frac{1}{\lambda} + \int_0^\infty P(\delta_t > x)\,dx$$

$$= \frac{1}{\lambda} + \int_0^t e^{-\lambda x}\,dx = \frac{1}{\lambda} + \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right) \xrightarrow[t \to \infty]{} \frac{2}{\lambda}$$

Excess life and current life of PP (cont.)

- Joint distribution of (γ_t, δ_t)

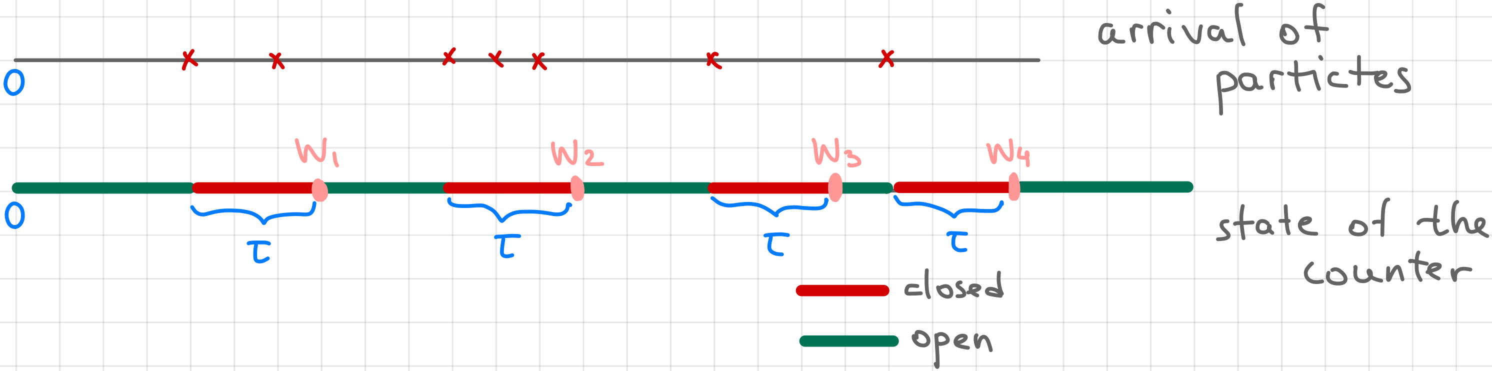
$$P(\gamma_t > x, \delta_t > y) = \begin{cases} 0, & \text{if } y > t \\ P(N(t+x) = N(t), N(t-y) = N(t)) = e^{-\lambda(x+y)}, & y < t \end{cases}$$

$\Rightarrow$ γ_t and δ_t are independent r.v.s for (PP)

Other renewal processes

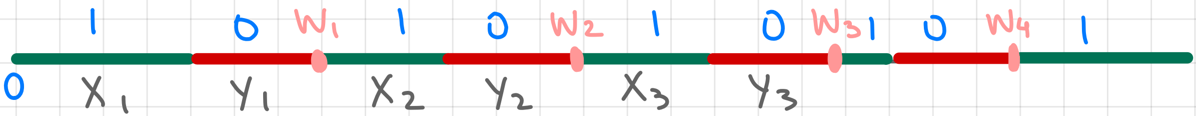
- traffic flow : distances between successive cars are assumed to be i.i.d. random variables

- counter process: particles/signals arrive on a device and lock it for time τ ; particles arrive according to a PP; times at which the counter unlocks form a renewal process



Other renewal processes

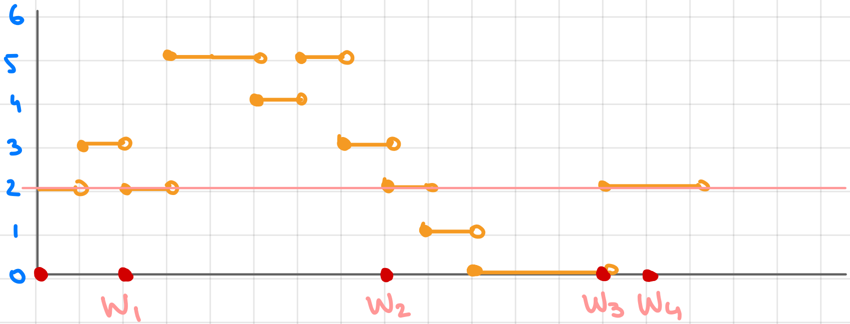
- more generally, if a component has two states (0/1, operating/non-operating etc), switches between them, times spent in 0 are X_i, times spent in 1 are Y_i, $(X_i)_{i=1}^{\infty}$ i.i.d., $(Y_i)_{i=1}^{\infty}$ i.i.d., then the times of switching from 0 to 1 form a renewal process with interrenewal times $X_i + Y_i$



Other renewal processes

- Markov chains: if $(Y_n)_{n\geq 0}$, $Y_n \in \{0,1,\dots\}$ is a recurrent MC starting from $Y_0 = k$, then the times of returns to state k form a renewal process. More precisely define

$$W_1 = \min\{n>0 : Y_n = k\}$$
$$W_p = \min\{n > W_{p-1} : Y_n = k\}$$



Example with $k=2$

Similarly for continuous time MCs.

Strong Markov property!

Other renewal processes

- Queues. Consider a single-server queueing process

(i) if customer arrival times form a renewal process then the times of the starts of successive idle periods generate a second renewal time

(ii) if customes arrive according to a Poisson process, then the times when the server passes from busy to free form a renewal process

Asymptotic behavior

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i, $X_i \in (0, \infty)$.

Thm.

$$P\left(\lim_{t\to\infty} N(t) = +\infty\right) = 1$$

Proof. $N(t)$ is nondecreasing, therefore $\exists \lim_{t\to\infty} N(t) =: \underset{(0,+\infty)\cup\{+\infty\}}{N_\infty}$

N_∞ is the total number of events ever happened.

$N_\infty \le k$ if and only if $W_{k+1} = \infty$

if and only if $X_i = \infty$ for some $1 \le i \le k+1$

$$P(N_\infty < \infty) = P(X_i = \infty \text{ for some } 1 \le i \le k+1) = P\left(\bigcup_{i=1}^{k+1} \{X_i = \infty\}\right) \le \sum_{i=1}^{k+1} P(X_i = \infty) = 0$$

Thm (Pointwise renewal thm).