MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- HW5 due Friday, May 12 on Gradescope

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$$P(\delta_t \geq x, \gamma_t > y) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} \big(1 - F(t+y-u)\big)\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} \big(1 - F(t+y-u)\big)\, d\overbrace{\sum_{k=1}^{\infty} F^{*k}(u)}^{M(t)}$$

$$= 1 - F(t+y) + \int_0^{t-x} \big(1 - F(\overbrace{t+y-u}^{=w})\big)\, m(u)\, du$$

$$= 1 - F(t+y) + \int_{x+y}^{y+t} \big(1 - F(w)\big)\, m(t+y-w)\, dw$$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ $(\mu = E(X_1))$. Then

$$\lim_{t\to\infty} P(\delta_t \geq x, \gamma_t > y) = \lim_{t\to\infty} \left(1 - F(t+y) + \int_{x+y}^{y+t} \big(1 - F(w)\big)\left\{\frac{1}{\mu} + \varepsilon(t+y-w)\right\} dw \right)$$

$$= \int_{x+y}^{\infty} \big(1 - F(w)\big)\frac{1}{\mu}\, dw + \lim_{t\to\infty} \int_{x+y}^{y+t} \big(1 - F(w)\big)\, \varepsilon(t+y-w)\, dw \to 0 \quad \text{Exercise}$$

Joint/limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

(a) $$P(\gamma_t > y, \delta_t \geq x) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$
$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, dM(u)$$

(b) if additionally the interrenewal times are continuous,
$$\lim_{t \to \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w))\, dw \qquad (*)$$

If we denote by $(\gamma_\infty, \delta_\infty)$ a pair of r.v.s with distribution $(*)$ then γ_∞ and δ_∞ are continuous r.v.s with densities

$$f_{\gamma_\infty}(x) = f_{\delta_\infty}(x) = \frac{1}{\mu}\left(1 - F(x)\right)$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on [0,1] (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

$$\lim_{t \to \infty} P\left(\gamma_t < \tfrac{1}{2}\right) = \int_0^{\frac{1}{2}} 2 \cdot (1-x)\, dx = 1 - x^2 \Big|_0^{1/2} = \frac{3}{4} = 0.75$$

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

$$\lim_{t \to \infty} P\left(\delta_t < \tfrac{1}{2}\right) = \int_0^{\frac{1}{2}} 2 \cdot (1-x)\, dx = 0.75$$

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{\infty} g(t-x) f(x)\, dx$

Def. Function h is called locally bounded if $\max_{0 \le x \le t} |h(x)| < \infty \ \forall t$

Def. Function h is absolutely integrable if

$$\int_0^{\infty} |h(x)|\, dx < \infty$$

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded

and $$H = h + H * F \quad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$,

then $$H = h + h * M \quad (**)$$ [convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Examples

- Renewal function: $M(t)$ satisfies

and
$$\underset{H(t)}{\underset{\parallel}{M(t)}} = \underset{h(t)}{\underset{\parallel}{F(t)}} + \underset{H(t)}{\underset{\parallel}{M}} * F(t) = \underset{h(t)}{\underset{\parallel}{F(t)}} + \underset{h * H(t)}{F * M(t)}$$

$F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

$$m(t) = f(t) + m * F(t)$$

and
$$= f(t) + f * M(t) \text{ (in the Riemann - Stieltjes sense)}$$

f is absolutely integrable, $\int_0^\infty f(x)\,dx = 1$, so

$$\lim_{t \to \infty} m(t) = \frac{1}{\mu} \int_0^\infty f(x)\,dx = \frac{1}{\mu}$$

Important remark

Let $W = (W_1, W_2, \dots)$ be renewal times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W' is independent of $W_1 = X_1$
- W' has the same distribution as W

Example

<u>Example</u>. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then $\gamma_t = X_1 - t$; if $X_1 < t$ condition on $X_1 = s$

$$E(\gamma_t) = E(\gamma_t \mathbb{1}_{X_1 > t}) + E(\gamma_t \mathbb{1}_{X_1 \leq t})$$

$$E(\gamma_t \mathbb{1}_{X_1 \leq t}) =$$

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