MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- HW6 due Monday, May 22 on Gradescope
- Midterm 2

Remark. $M(t)$ is finite for all t

Proposition. Let $N(t)$ be a renewal process with interrenewal times X_i having distribution F. If there exist $c>0$ and $\alpha\in(0,1)$ such that $P(X_1>c)>\alpha$, then $M(t)=E(N(t))<\infty \ \forall t$

Proof: Recall that $M(t)=\sum_{k=1}^{\infty}P(W_k\le t)=\sum_{k=1}^{\infty}P\left(\sum_{j=1}^{k}X_j\le t\right)$ (*)

Fix $t>0$. Take $L\in\mathbb{N}$ $c\cdot L>t$. Then

$$P\left(\sum_{j=1}^{L}X_j>t\right)\geq P(X_1>c, X_2>c, \dots, X_L>c)>\alpha^L>0$$

$$P\left(\sum_{j=1}^{L}X_j\le t\right)\le 1-\alpha^L<1.$$ Thus, for any $n\in\mathbb{N}$

$$P(W_{nL}\le t)=P\left(\sum_{j=1}^{nL}X_j\le t\right)\le P\left(\sum_{j=1}^{L}X_j\le t, \sum_{j=L+1}^{2L}X_j\le t, \dots\right)\le(1-\alpha^L)^n$$

From this (exercise) we conclude that $\sum_{k=1}^{\infty}P(W_k\le t)=M(t)<\infty.$ ∎

Example: Age replacement policies (PK, p. 363)

Setting: - component's lifetime has distribution function F

- component is replaced

  (A) either when it fails,

  (B) or after reaching age T (fixed)

  whichever occurs first

- replacements (A) and (B) have different costs:

  replacement of a failed component (A) is more expensive than the planned replacement (B)

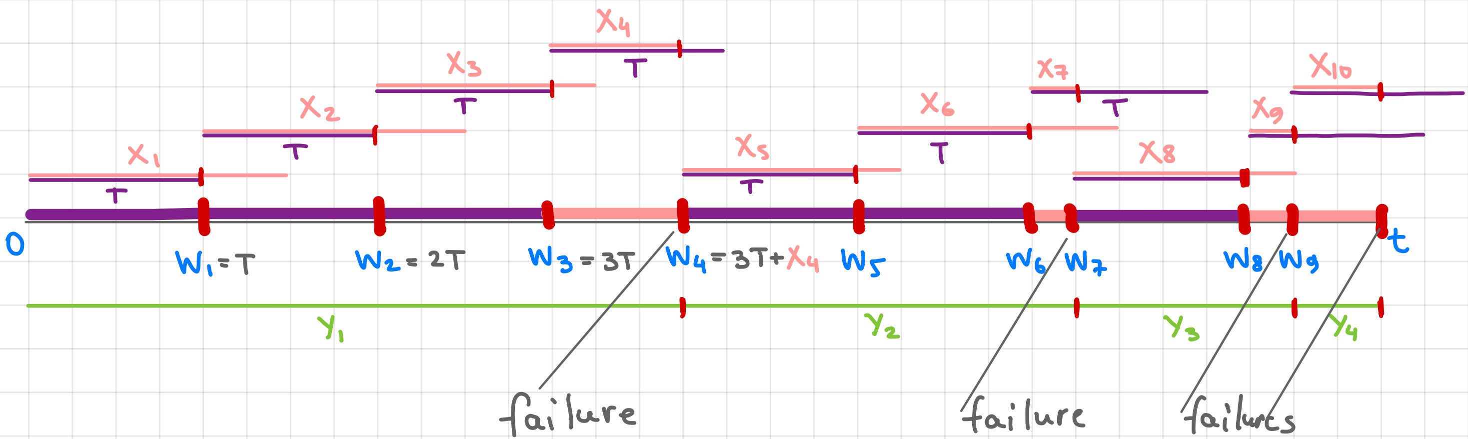
Question: How does the long-run cost of replacement depend on the cost of (A), (B) and age T?

What is the optimal T that minimizes the long-run cost of replacement?

Example: Age replacement policies (PK, p. 363)

Notation: X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures



Here we have two renewal processes

(1) renewal process $N(t)$ generated by renewal times $(W_i)_{i=1}^{\infty}$

(2) renewal process $Q(t)$ generated by interrenewal times $(Y_i)_{i=1}^{\infty}$

$N(t) = \#$ replacements on $[0,t]$, $Q(t) = \#$ failure replacements on $[0,t]$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $N(t)$

$$W_i - W_{i-1} = \begin{cases} X_i, & \text{if } X_i \leq T \\ T, & \text{if } X_i > T \end{cases}, \quad \text{so}$$

$$\int_0^\infty \underbrace{P(X > x)}_{1-F(x)} dx = E(X)$$

$$F^{(T)}(x) := P(W_i - W_{i-1} \leq x) = \begin{cases} F(x), & \text{if } x \leq T \\ 1, & \text{if } x > T \end{cases}$$

In particular,

$$E(W_i - W_{i-1}) = \int_0^T (1 - F(x))\, dx =: \mu_T \leq \mu = E(X_i)$$

Using the elementary renewal theorem for $N(t)$, the total number of replacements has a long-run rate

$$\frac{E(N(t))}{t} \approx \frac{1}{\mu_T} \quad \text{for large } t$$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $Q(t)$.

$$Y_1 = \begin{cases} X_1 & \text{if } X_1 \le T \\ T + X_2 & \text{if } X_1 > T,\ X_2 \le T \\ \vdots & \\ nT + X_{n+1}, & \text{if } X_1 > T, \dots, X_n > T,\ X_{n+1} \le T \\ \vdots & \end{cases}$$

so $Y_1 = L \cdot T + Z$, where $P(L \ge n) = (1 - F(T))^n$, $Z \in [0,T]$

and for $z \in (0,T)$

$$P(Z \le z) = P(X_1 \le z,\ \cancel{X_1 \le T}) + P(X_2 \le z,\ X_1 > T,\ \cancel{X_2 \le T})$$

$$+ \cdots + P(X_{n+1} \le z,\ X_1 > T, \dots, X_n > T,\ \cancel{X_{n+1} \le T}) + \cdots$$

$$= P(X_1 \le z) + P(X_2 \le z) P(X_1 > T) + \cdots + P(X_{n+1} \le z) P(X_1 > T, X_2 > T, \dots, X_n > T) + \cdots$$

$$= F(z)\left(1 + (1 - F(T)) + (1 - F(T))^2 + \cdots + (1 - F(T))^n + \cdots\right) = \frac{F(z)}{F(T)}$$

Example: Age replacement policies (PK, p. 363)

Now we can compute the long-run rate of the replacements due to failures

$E(Y_1) = T\,E(L) + E(Z)$

$E(L) =$

.

$E(Z) =$, so

$E(Y_1) =$

Applying the elementary renewal theorem to $Q(t)$