MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: nothing

Week 10:

- homework 8 (due Friday, June 9)

Reflected BM

Def. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$R_t := |B_t| = \begin{cases} B_t, & \text{if } B(t) \geq 0 \\ -B_t, & \text{if } B(t) < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) = \int_{-\infty}^{+\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \sqrt{\frac{2t}{\pi}}$

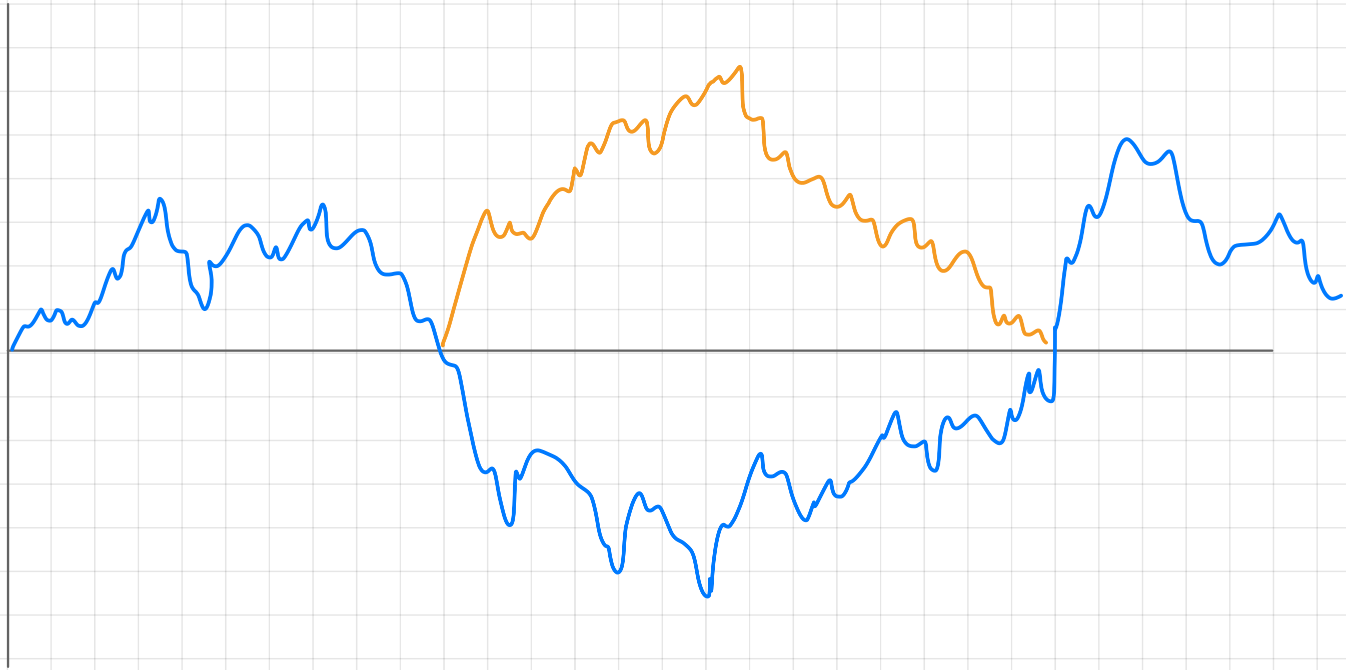
$$Var(R_t) = E(B_t^2) - (E(|B_t|))^2 = t - \left(\sqrt{\frac{2t}{\pi}}\right)^2 = \left(1 - \frac{2}{\pi}\right)t$$

Transition density: $P(R_t \leq y \mid R_0 = x) = P(-y \leq B_t \leq y \mid B_0 = x)$

$$= \qquad \Rightarrow p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}} \right)$$

Thm (Lévy) Let $M_t = \max_{0\leq u\leq t} B_u$. Then $(M_t - B_t)_{t\geq 0}$ is a reflected BM.

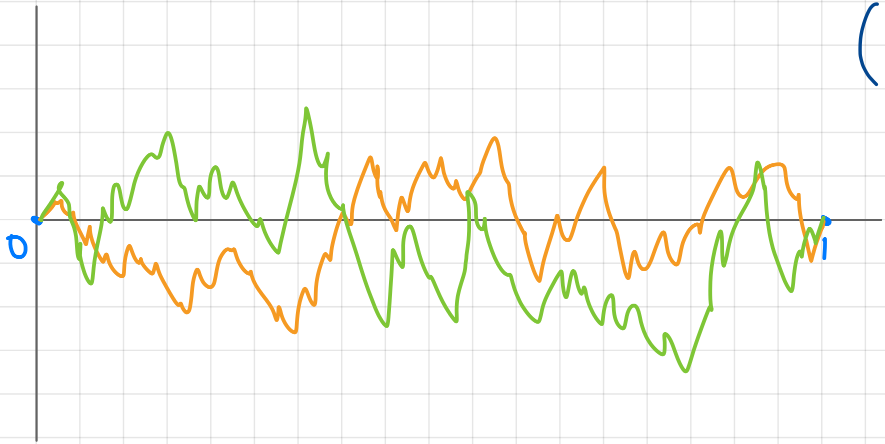
Reflected BM



Brownian bridge

Brownian bridge is constructed from a BM by conditioning on the event $\{B(0)=0, B(1)=0\}$.

$(B_t^\circ)_{0\le t\le 1}$



Thm 1. Brownian bridge is a continuous Gaussian process on $[0,1]$ with mean 0 and covariance function

$$\Gamma(s,t) = \min\{s,t\} - st$$

If $(B_t)_{t\ge 0}$ is a sBM, then $(B_t - tB_1)_{0\le t\le 1}$ is a BB

Brownian motion with drift

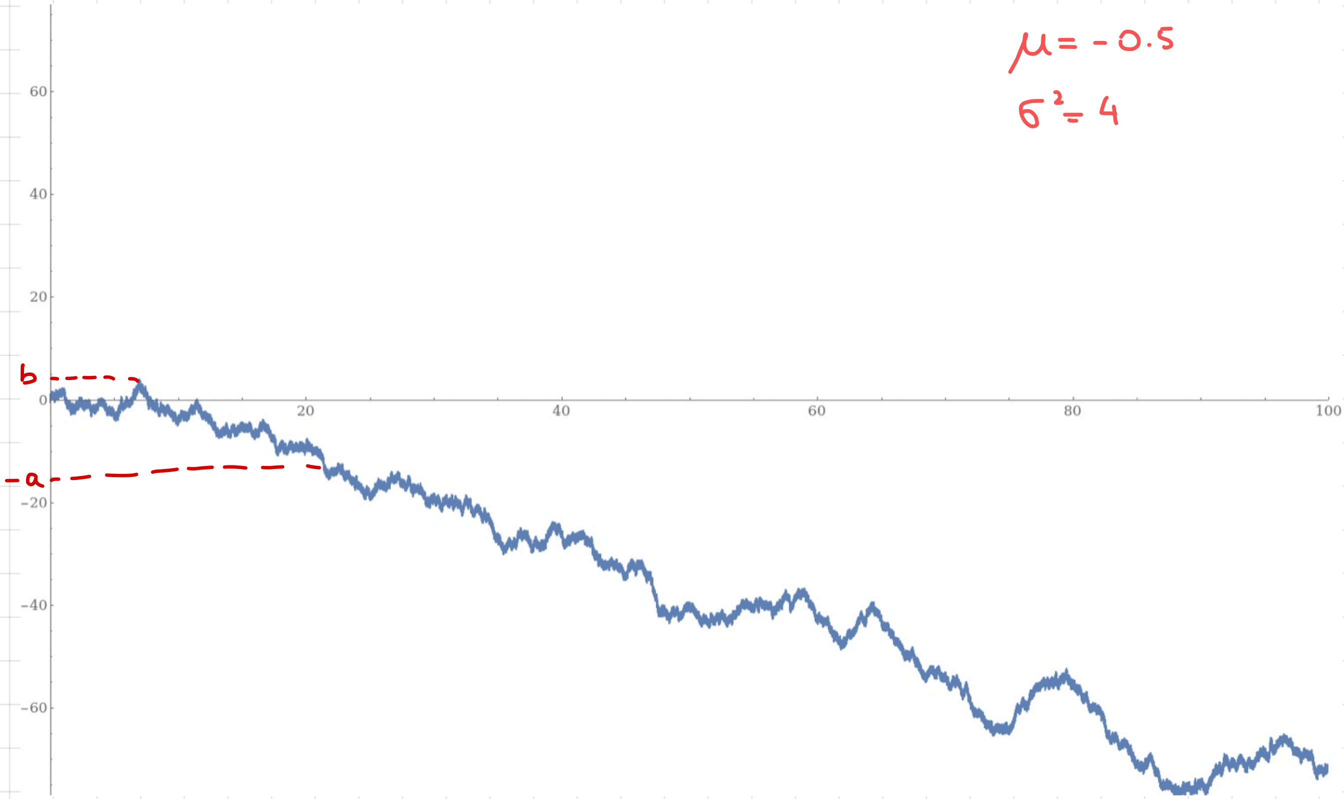
<u>Def</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then for $\mu \in \mathbb{R}$ and $\sigma > 0$ the process $(X_t)_{t\geq 0}$ with $X_t = \mu t + \sigma B_t$, $t \geq 0$ is called the Brownian motion with drift μ and variance paremeter σ^2.

<u>Remark</u> BM with drift μ and variance paremeter σ is a stochastic process $(X_t)_{t\geq 0}$ satisfying

1) $X_0 = 0$, $(X_t)_{t\geq 0}$ has continuous sample paths
2) $(X_t)_{t\geq 0}$ has independent increments
3) For $t > s$ $\quad X_t - X_s \sim N(\mu(t-s), \sigma^2(t-s))$

In particular, $X_t \sim N(\mu t, \sigma^2 t) \Rightarrow X_t$ is not centered, not symmetric w.r.t. the origin

Brownian motion with drift



Gambler's ruin problem for BM with drift

Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$. Fix $a < x < b$ and denote

$$T = T_{ab} = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}, \text{ and}$$

$$u(x) = P(X_T = b \mid X_0 = x).$$

Theorem.

(i) $$u(x) = \frac{\exp(-2\mu x/\sigma^2) - \exp(-2\mu a/\sigma^2)}{\exp(-2\mu b/\sigma^2) - \exp(-2\mu a/\sigma^2)}$$

(ii) $$E(T_{ab} \mid X_0 = x) = \frac{1}{\mu}\Big(u(x)(b-a) - (x-a)\Big)$$

No proof

$$\text{SBM: } u(x) = \frac{x-a}{b-a}$$

Example

Fluctuations of the price of a certain share is modeled by the BM with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

(a) What is the probability that you will sell at profit?

(b) What is the expected time until you sell the share?

Denote by $(X_t)_{t\geq 0}$ a BM with drift $\frac{1}{10}$ and variance 4, $x = 100$, $b = 110$, $a = 95$. Then $2\mu/\sigma^2 = \frac{2 \cdot 0.1}{4} = \frac{1}{20}$ and

(a) $P(X_T = 110 \mid X_0 = 100) = \dfrac{e^{-\frac{100}{20}} - e^{-\frac{95}{20}}}{e^{-\frac{110}{20}} - e^{-\frac{95}{20}}} \approx 0.419$

(b) $E(T \mid X_0 = 100) = \approx 12.88$

Maximum of a BM with negative drift

Thm Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu<0$, variance σ^2 and $X_0=0$. Denote $M=\max\limits_{t\geq 0} X_t$. Then

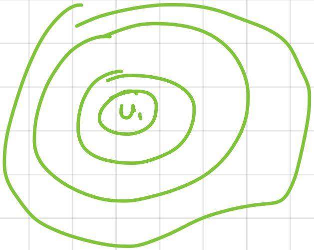
$$M \sim \mathrm{Exp}\left(-\frac{2\mu}{\sigma^2}\right)$$

$U_1 \subset U_2 \subset U_3 \subset \cdots$

$$P\left(\bigcup_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} P(U_n)$$

Proof. $X_0=0$, therefore $M\geq 0$. For any $b>0$

$$P(M>b) = P\left(\bigcup_{n=1}^{\infty}\{X \text{ hits } b \text{ before } -n\}\right)$$



$$= \lim_{n\to\infty} P(X \text{ hits } b \text{ before } -n)$$

$$= \lim_{n\to\infty} \frac{1 - \exp(+\cancelto{0}{2\mu n}/\sigma^2)}{\exp(-2\mu b/\sigma^2) - \exp(+\cancelto{0}{2\mu n}/\sigma^2)} = e^{2\mu b/\sigma^2}$$

$$P(M>b) = e^{\left(-\frac{2\mu}{\sigma^2} b\right)}$$

$$\Rightarrow \quad M \sim \mathrm{Exp}\left(-\frac{2\mu}{\sigma^2}\right)$$

Geometric BM

Def. Stochastic process $(Z_t)_{t\geq 0}$ is called a geometric Brownian motion with drift parameter α and variance σ^2 if $X_t = \log(Z_t)$ is a BM with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance σ^2.

In other words, $Z_t = z\, e^{\sigma B_t + \alpha t - \frac{1}{2}\sigma^2 t}$, where $(B_t)_{t\geq 0}$ is a standard BM and $z > 0$ is the starting point $Z_0 = z$.

If $0 \leq t_1 < t_2 < \dots < t_n$, then $\frac{Z_{t_i}}{Z_{t_{i-1}}} = e^{(\alpha - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma(B_{t_i} - B_{t_{i-1}})}$

Since B has independent increments

$\frac{Z_{t_1}}{Z_{t_0}}, \frac{Z_{t_2}}{Z_{t_1}}, \dots, \frac{Z_{t_n}}{Z_{t_{n-1}}}$ are independent and

$$\frac{Z_{t_n}}{Z_{t_0}} = \frac{Z_{t_1}}{Z_{t_0}} \cdot \frac{Z_{t_2}}{Z_{t_1}} \cdots \frac{Z_{t_n}}{Z_{t_{n-1}}}$$ ⟵ "relative change of price = product of independentent relative changes"

Expectation of Geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ.

Then

$$E(Z_t \mid Z_0 = z) = E\left(z \cdot e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t}\right) = z\, e^{(\alpha - \frac{1}{2}\sigma^2)t} E\left(e^{\sigma B_t}\right)$$

$$E(e^{\sigma B_t}) = e^{\frac{\sigma^2 t}{2}}$$

$$\Rightarrow E(Z_t \mid Z_0 = z) = z\, e^{(\alpha - \frac{1}{2}\sigma^2)t} e^{t\frac{\sigma^2}{2}} = z\, e^{\alpha t}$$

Remark

It can be shown that for $0 < \alpha < \frac{1}{2}\sigma^2$ $Z_t \to 0$ as $t \to \infty$

At the same time, for $\alpha > 0$ $E(Z_t) \to \infty$.

Variance of geometric BM

$$E(Z_t^2 \mid Z_0 = z) =$$

$$=$$

$$Var(Z_t \mid Z_0 = z) =$$

Theorem.

Let $(Z_t)_{t \geq 0}$ be geometric BM with paremeters α and σ^2.

Then

(i) $E(Z_t \mid Z_0 = z) = z e^{\alpha t}$

(ii) $Var(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t}(e^{\sigma^2 t} - 1)$

Gambler's ruin for geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.
Let $A<1<B$, and denote $T = \min\{t : \frac{Z_t}{Z_0} = A \text{ or } \frac{Z_t}{Z_0} = B\}$.

Theorem

$$P\left(\frac{Z_T}{Z_0} = B\right) = \frac{1 - A^{1-\frac{2\alpha}{\sigma^2}}}{B^{1-\frac{2\alpha}{\sigma^2}} - A^{1-\frac{2\alpha}{\sigma^2}}}$$

Example Fluctuations of the price are modeled by a geometric BM with drift $\alpha = 0.1$ and variance $\sigma^2 = 4$. You buy a share at 100\$ and plan to sell it if its price increases to 110\$ or drops to 95\$.

Take $A = 0.95$, $B = 1.1$, $2\alpha/\sigma^2 = \frac{1}{20}$, $1 - 2\alpha/\sigma^2 = \frac{19}{20} = 0.95$

$$P(X_T = 110 \mid X_0 = 100) = \frac{1 - 0.95^{0.95}}{1.1^{0.95} - 0.95^{0.95}} \approx 0.334$$