MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Hitting probabilities.
Absorption times. General CTMC.
Matrix exponentials
Next: PK 6.5, 6.6, Durrett 4.1

Week 2:

- HW1 due Friday, April 14 on Gradescope
- Important: Midterm 1 will take place on Friday, April 28

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Strong Markov property

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = i$,

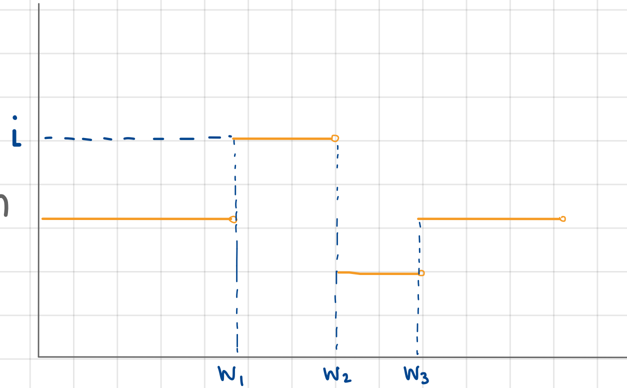
$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

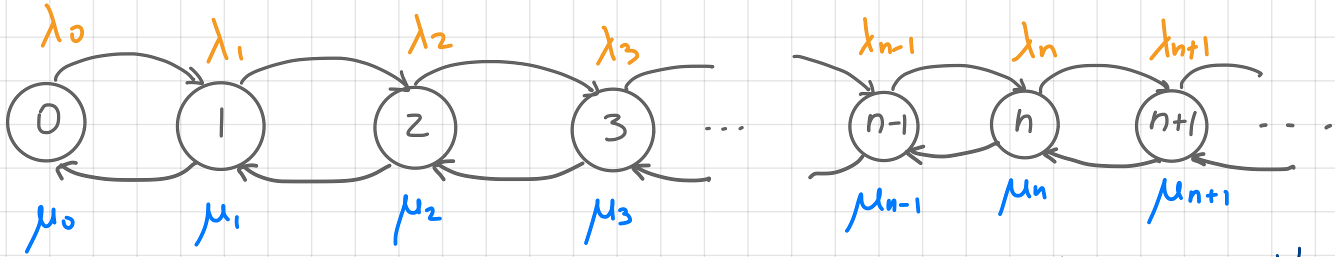
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

Example

$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0 = i$ and is indep. of what happened before



Alternative (jump and hold) characterization



$\lambda = \mu = 1 \qquad \lambda' = \mu' = 2$

Sojourn times S_k are independent, $\quad \frac{\lambda}{\lambda+\mu} = \frac{\lambda'}{\lambda'+\mu'} = \frac{1}{2}$

$\text{Exp}(2) \qquad \text{Exp}(4)$

Each transition has two parts

- wait in state i for time $\sim \text{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\rightarrow$ $i+1$ with probability $\dfrac{\lambda_i}{\lambda_i + \mu_i}$

go $i-1$ $\leftarrow$ with pobability $\dfrac{\mu_i}{\lambda_i + \mu_i}$

Alternative (jump and hold) characterization

"Proof" cont. $G_i(t) = P(S_k > t \mid X_{W_k} = i)$

$G_i'(t) = -(\lambda_i+\mu_i)G_i(t)$, $G_i(0)=1$

$\hookrightarrow G_i(t) = e^{-(\lambda_i+\mu_i)t} = P(S_k>t \mid X_{W_k}=i)$

✓ $\hookrightarrow S_k \sim \mathrm{Exp}(\lambda_i+\mu_i)$ (given that the process sojourns in i)

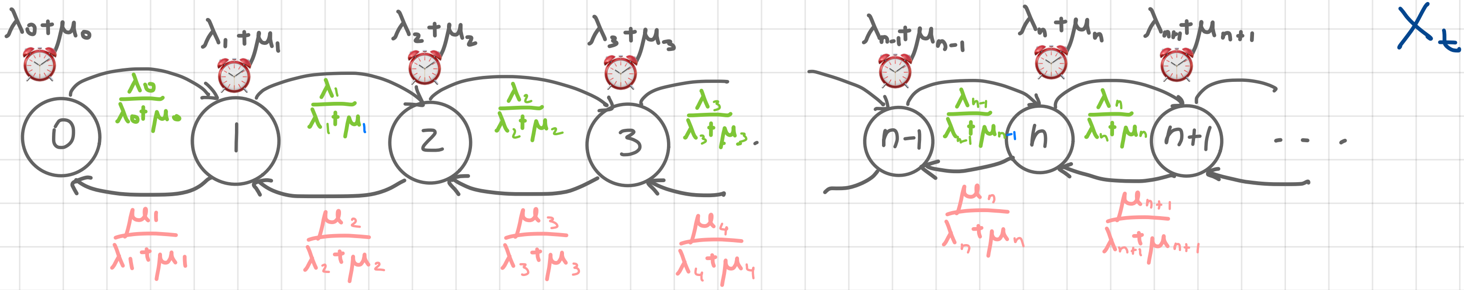
Suppose the process waits $\mathrm{Exp}(\lambda_i+\mu_i)$, then jumps to $i+1$ with probability $\lambda_i/(\lambda_i+\mu_i)$
to $i-1$ with probability $\mu_i/(\lambda_i+\mu_i)$

$P_{i,i+1}(h)$ "=" $P(S_0 \le h, \text{ jump to } i+1 \mid X_{W_0}=i) + o(h)$

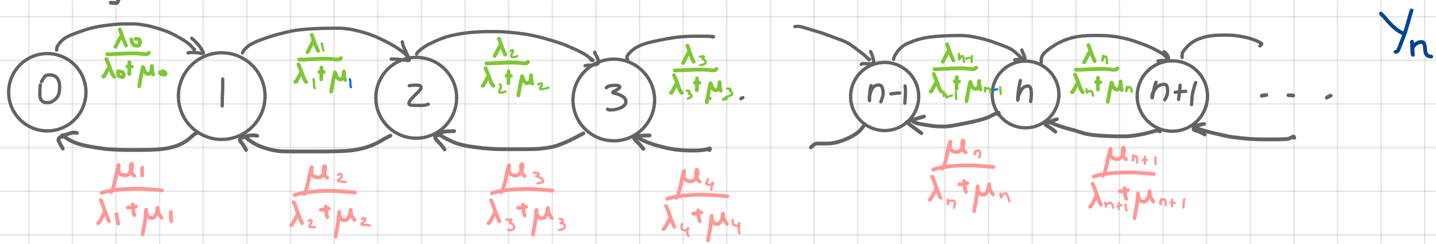
$$= \left(1-e^{-(\lambda_i+\mu_i)h}\right)\frac{\lambda_i}{\lambda_i+\mu_i} + o(h) = \left(1-\left(1-(\lambda_i+\mu_i)h \overset{+o(h)}{}\right)\right)\frac{\lambda_i}{\lambda_i+\mu_i} + o(h) = \lambda_i h + o(h)$$ ✓

$P_{i,i-1}(h)$ "=" $(\lambda_i+\mu_i)h \cdot \frac{\mu_i}{\lambda_i+\mu_i} + o(h) = \mu_i h + o(h)$

Related discrete time MC.



<u>Def</u>. Let $(X_t)_{t\geq 0}$ be a continuous time MC, let $W_n, n\geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n\geq 0}$ defined by $Y_n = X_{W_n}$ the jump chain of $(X_t)_{t\geq 0}$.



↰ random walk

Absorption probabilities for B&D processes

Let $(X_t)_{t\geq 0}$ be a birth and death process, and assume that the state 0 is absorbing, $\lambda_0 = 0$. Then

$$P((X_t)_{t\geq 0} \text{ gets absorbed in } 0 \mid X_0 = i)$$

$$= P((Y_n)_{n\geq 0} \text{ gets absorbed in } 0 \mid Y_0 = i)$$

↳ use the first step analysis to compute the absorption probabilities for $(Y_n)_{n\geq 0}$ (and for $(X_t)_{t\geq 0}$)

Denote $u_i = P((Y_n) \text{ is absorded in } 0 \mid Y_0 = i)$

Then $\quad u_0 = 1, \quad u_n = \dfrac{\mu_n}{\lambda_n + \mu_n} \cdot u_{n-1} + \dfrac{\lambda_n}{\lambda_n + \mu_n} \cdot u_{n+1}$

Absorption probabilities for B&D processes

$$u_0 = 1 , \qquad u_n = \frac{\mu_n}{\lambda_n+\mu_n} u_{n-1} + \frac{\lambda_n}{\lambda_n+\mu_n} u_{n+1}$$

Rewrite $\quad (\lambda_n + \mu_n) u_n = \mu_n u_{n-1} + \lambda_n u_{n+1}$

$$\lambda_n (u_{n+1} - u_n) = \mu_n (u_n - u_{n-1})$$

$$u_{n+1} - u_n = \frac{\mu_n}{\lambda_n}(u_n - u_{n-1})$$

$$= \underbrace{\frac{\mu_n}{\lambda_n} \cdot \frac{\mu_{n-1}}{\lambda_{n-1}} \cdot \cdots \cdot \frac{\mu_1}{\lambda_1}}_{\rho_n} (u_1 - \overset{=1}{u_0})$$

$$(*) \quad u_{n+1} - u_n = \rho_n (u_1 - 1)$$

Note that $\sum_{k=1}^{n-1} (u_{k+1} - u_k) = u_n - u_1 = (u_1 - 1) \sum_{k=1}^{n-1} \rho_k \qquad (**)$

If $\sum_{n=1}^{\infty} \rho_n = \infty$, then $u_1 = 1$ and from $(*)$ $u_n = 1 \ \forall n \geq 0$.

Absorption probabilities for B&D processes

Let $\sum_{k=1}^{\infty} \rho_k < \infty$. We are looking for the minimal solution that satisfies $u_n \in [0,1]$ $\forall n$. We rewrite (**) as

$$u_n = u_1 + (u_1 - 1)\sum_{k=1}^{n-1} \rho_k = 1 + (u_1 - 1)\left(1 + \sum_{k=1}^{n-1} \rho_k\right)$$

Choose smallest $u_1 \in [0,1]$ for which $1 + (u_1 - 1)\left(1 + \sum_{k=1}^{n-1} \rho_k\right) \geq 0$ $\forall n$

$$u_1 \geq 1 - \frac{1}{1 + \sum_{k=1}^{n-1} \rho_k} \geq 1 - \frac{1}{1 + \sum_{k=1}^{\infty} \rho_k}, \quad u_1 = \frac{\sum_{k=1}^{\infty} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$

$$u_n = \frac{\sum_{k=n}^{\infty} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$